Elementary Theory and Methods for Elliptic Partial Differential Equations

John Villavert

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CHAPTER 1

Introduction and Basic Theory

In this introductory chapter, we provide some preliminary background which we will use later in establishing various results for general elliptic partial differential equations (PDEs). The material found within these notes aims to compile the fundamental theory for second-order elliptic PDEs and serves as complementary notes to many well-known references on the subject, c.f., [5, 6, 8, 11, 13]. Several recommended resources on basic background that supplement these notes and the aforementioned references are the textbooks [2, 9, 21].

We will mainly focus on the Dirichlet problem,

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{1.1}$$

where U is a bounded open subset of $\mathbb{R}^n$ with boundary ∂U, and $u : \mathbb{R}^n \mapsto \mathbb{R}$ is the unknown quantity. For this problem, $f : U \mapsto \mathbb{R}$ is given, and L is a second-order differential operator having either the form

$$Lu = -\sum_{i,j=1}^{n} D_j \left(a^{ij}(x) D_i u \right) + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u, \tag{1.2}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u, \tag{1.3}$$

for given coefficient functions a^{ij}, b^i, and c ($i, j = 1, 2, \ldots, n$) which are assumed to be measurable in $\bar{U}$, the closure of the set U. However, in this chapter, we take these coefficients to be continuous in $\bar{U}$. If L takes the form (1.2), then it is said to be in **divergence form**, and if it takes the form (1.3), then it is said to be in **non-divergence form**.

Remark 1.1. *Here, $D_{ij} = D_i D_j$. In practice,* (1.2) *is natural for energy methods while (1.3) is more appropriate for the maximum principles. In addition, the Dirichlet problem* (1.1) *can be extended to systems, i.e., $Lu_i = f_i$ in U, and $u_i = 0$ on ∂U, for $i = 1, 2, \dots, L \in \mathbb{Z}^+$. A simple example of a second-order differential operator is the Laplacian, $L := -\Delta$, where $a^{ij} = \delta_{ij}$, $b^i = c = 0$ $(i, j = 1, 2 \dots, n)$ in either* (1.2) *or* (1.3).

Remark 1.2. *The elliptic theory for equations in divergence form was developed first as we can easily exploit the distributional framework and energy methods for weak solutions in Sobolev spaces, for example. Much of our focus in these notes will be on establishing the basic elliptic PDE theory for equations in divergence form.*

Remark 1.3. *Extending this theory to elliptic equations in non-divergence form has certain obstacles, and its treatment requires a somewhat different approach. We shall study one way of examining such equations using another concept of a weak solution called a viscosity solution, which are defined with the help of maximum and comparison principles. We shall give a brief introduction to fully nonlinear elliptic equations in non-divergence form and their viscosity solutions in Chapter 4.*

Unless stated otherwise, we shall always assume that L is **uniformly elliptic**, i.e., there exist $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ a.e } x \in U, \text{ for all } \xi \in \mathbb{R}^n.$$

Moreover, $u \in H_0^1(U)$ is said to be a *weak solution* of (1.1) in divergence form if

$$B[u,\, v] = (f,\, v), \text{ for all}\, v \in H_0^1(U),$$

where $B[\cdot,\, \cdot]$ is the associated bilinear form,

$$B[u,\, v] := \int_U \sum_{i,j=1}^{n} a^{ij} D_i u D_j v + \sum_{i=1}^{n} b^i(x) D_i u v + c(x) u v \, dx.$$

1.1 Harmonic Functions

First we shall introduce the mean-value property, which provides the key ingredient in establishing many important properties for harmonic functions.

1.1.1 Mean Value Properties

Definition 1.1. *For $u \in C(U)$ we define* *(i) u satisfies the first mean value property (in U) if*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y \quad \textit{for any } B_r(x) \subset U;$$

(ii) u satisfies the second mean value property if

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy \quad \textit{for any } B_r(x) \subset U.$$

Remark 1.4. *These two definitions are equivalent. To see this, observe that if we rewrite (i) as*

$$u(x) r^{n-1} = \frac{1}{\omega_n} \int_{\partial B_r(x)} u(y)\, d\sigma_y,$$

where ω_n denotes the surface area of the $(n-1)$-dimensional unit sphere $\mathcal{S}^n$, then integrate with respect to r, we get

$$u(x)\frac{r^n}{n} = \frac{1}{\omega_n} \int_0^r \int_{\partial B_s(x)} u(y)\, d\sigma_y\, ds = \frac{1}{\omega_n} \int_{B_r(x)} u(y)\, dy.$$

If we rewrite (ii) as

$$u(x) r^n = \frac{n}{\omega_n} \int_{B_r(x)} u(y)\, dy = \frac{n}{\omega_n} \int_0^r \int_{\partial B_s(x)} u(y)\, d\sigma_y ds$$

then differentiate with respect to r, we obtain (i).

Remark 1.5. *The mean value properties can easily be expressed in the following ways.*

(i) $u \in C(U)$ satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x + r\omega)\, d\sigma_\omega \quad \textit{for any } B_r(x) \subset U;$$

(ii) $u \in C(U)$ satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n} \int_{B_1(0)} u(x + ry)\, dy \quad \textit{for any } B_r(x) \subset U;$$

Theorem 1.1. *If $u \in C^2(U)$ is harmonic, then u satisfies the mean value property.*

Proof. Set

$$\phi(r) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x + r\omega)\, d\sigma_\omega.$$

Then

$$\begin{aligned}\phi'(r) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} Du(x+r\omega)\cdot\omega\, d\sigma_\omega = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} Du(y)\cdot\frac{y-x}{r}\, d\sigma_y \\ &= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)}\frac{\partial u}{\partial\nu}(y)\, d\sigma_y = \frac{r}{n}\frac{n}{\omega_n r^n}\int_{\partial B_r(x)}\frac{\partial u}{\partial\nu}(y)\, d\sigma_y \\ &= \frac{r}{n}\frac{1}{|B_r(x)|}\int_{B_r(x)}\Delta u(y)\, dy = 0.\end{aligned}$$

Hence, ϕ is constant. Therefore, by the Lebesgue differentiation theorem (see Theorem 3.4),

$$\phi(r) = \lim_{t\longrightarrow 0}\phi(t) = \lim_{t\longrightarrow 0}\frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} u(y)\, d\sigma_y = u(x).$$

$\square$

The next theorem is the converse of the previous result. Namely, functions satisfying the mean value property are harmonic.

Theorem 1.2. *If $u \in C^2(U)$ satisfies the mean value property, then u is harmonic.*

Proof. If $\Delta u \not\equiv 0$, we may assume without loss of generality that there exists a ball $B_r(x) \subset U$ for which $\Delta u > 0$ within $B_r(x)$ However, as in the previous computation,

$$0 = \phi'(r) = \frac{r}{n}\frac{1}{|B_r(x)|}\int_{B_r(x)}\Delta u(y)\, dy > 0,$$

which is a contradiction. $\square$

The next theorem is the maximum principle for harmonic functions.

Theorem 1.3 (Strong maximum principle for harmonic functions)**.** *Suppose $u \in C^2(U)\cap C(\bar{U})$ is harmonic within U.*

(i) Then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(ii) In addition, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u(x),$$

then u is constant in U.

Proof. Suppose that there is such a point $x_0 \in U$ with $u(x_0) = M := \max_{\bar{U}} u$. Then for $0 < r < dist(x_0, \partial U)$, the mean value property asserts

$$M = u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(y)\, dy \le M.$$

Hence, equality holds only if $u \equiv M$ in $B_r(x_0)$. That is, the set $\{x \in U \mid u(x) = M\}$ is both open and relatively closed in U. Therefore, this set must equal U since U is connected. This proves assertion (ii), from which (i) follows. $\square$

1.1.2 Sub-harmonic and Super-harmonic Functions

Interestingly, mean-value properties and maximum principles hold for sub-harmonic and super-harmonic functions. Let us state such results including some important applications. We say a function $u \in C^2(U)$ is sub-harmonic in U if $-\Delta u \leq 0$ in U and super-harmonic if $-\Delta u \geq 0$ in U.

Lemma 1.1 (Mean Value Inequality). *Let $x \in B_{r_0}(x) \subset U$ for some $r_0 > 0$.*

(i) If $-\Delta u > 0$ within $B_{r_0}(x)$, then for any $r \in (0, r_0)$,

$$u(x) > \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y.$$

It follows that if x_0 is a minimum point of u in U, then

$$-\Delta u(x_0) \leq 0.$$

(ii) If $-\Delta u < 0$ within $B_{r_0}(x)$, then for any $r \in (0, r_0)$,

$$u(x) < \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, d\sigma_y.$$

It follows that if x_0 is a maximum point of u in U, then

$$-\Delta u(x_0) \geq 0.$$

Proof. As in the proof of Theorem 1.1, we see

$$\int_{B_r(x)} \Delta u(x)\, dx = r^{n-1} \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x + r\omega)\, d\sigma_\omega. \tag{1.4}$$

We only prove (i) since the proof of (ii) follows from similar arguments. From (1.4), we see that if $-\Delta u > 0$, then

$$\frac{\partial}{\partial r} \int_{\partial B_1(0)} u(x + r\omega)\, d\sigma_\omega < 0.$$

Integrating this from 0 to r yields

$$\int_{\partial B_1(0)} u(x + r\omega)\, d\sigma_\omega - u(x)|\partial B_1(0)| < 0,$$

in which the desired inequality follows immediately. To prove the second statement in (i), we proceed by contradiction. On the contrary, suppose that x_0 is a minimum point of u in U and assume that $-\Delta u(x_0) > 0$. By the continuity of u, we can find a $\delta > 0$ for which $-\Delta u > 0$ within $B_\delta(x_0)$. But the mean value inequality implies that

$$u(x_0) > \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u(y)\, d\sigma_y \quad \text{for any} \quad r \in (0, \delta).$$

This contradicts with the assumption that x_0 is a minimum of u. $\square$

A nice application of the mean value inequalities is the weak maximum principle for the Laplacian. Analogous results for more general uniformly elliptic equations are provided below. In addition, unlike the strong maximum principles for harmonic functions provided earlier, we do not make any connectedness assumption on the domain U.

Theorem 1.4 (Weak Maximum Principle for the Laplacian). *Suppose that* $u \in C^2(U) \cap C(\bar{U})$.

(i) If

$$-\Delta u \geq 0 \quad \textit{within} \quad U,$$

then

$$\min_{\bar{U}} u \geq \min_{\partial U} u.$$

(ii) If

$$-\Delta u \leq 0 \quad \textit{within} \quad U,$$

then

$$\max_{\bar{U}} u \leq \max_{\partial U} u.$$

Proof. We only prove (i) since (ii) follows from similar arguments. First, we assume u is strictly super-harmonic: $-\Delta u > 0$ within U. Let x_0 be a minimum of u in U, but the mean value inequality implies $-\Delta u(x_0) \leq 0$, which is a contradiction. Thus, $\min_{\bar{U}} u \geq \min_{\partial U} u$. Now, suppose u is super-harmonic: $-\Delta u \geq 0$ within U and set $u_\epsilon = u - \epsilon|x|^2$. Obviously, u_ϵ is strictly super-harmonic, i.e.,

$$-\Delta u_\epsilon = -\Delta u + 2\epsilon n > 0.$$

It follows that $\min_{\bar{U}} u_\epsilon \geq \min_{\partial U} u_\epsilon$ and the desired result follows after sending $\epsilon \longrightarrow 0$. $\square$

An application of the weak maximum principle is the following interior gradient estimate for harmonic functions.

Corollary 1.1 (Bernstein). *Suppose u is harmonic in U and let $V \subset\subset U$. Then there holds*

$$\sup_V |Du| \leq C \sup_{\partial U} |u|,$$

where $C = C(n, V)$ is a positive constant. In particular, for any $\alpha \in (0,1)$ there holds

$$|u(x) - u(y)| \leq C|x-y|^\alpha \sup_{\partial U} |u| \quad \textit{for any} \quad x, y \in V.$$

Proof. A direct calculation shows

$$\Delta(|Du|^2) = 2\sum_{i,j=1}^{n}(D_{ij}u)^2 + 2\sum_{i=1}^{n} D_iuD_i(\Delta u) = 2\sum_{i,j=1}^{n}(D_{ij}u)^2 \geq 0. \tag{1.5}$$

That is, $|Du|^2$ is a sub-harmonic function in U. Then, for any test function $\varphi \in C_0^1(U)$, a basic identity yields

$$\Delta(\varphi|Du|^2) = (\Delta\varphi)|Du|^2 + 2D\varphi \cdot D(|Du|^2) + \varphi\Delta(|Du|^2).$$

Hence, combining this with (1.5) gives us

$$\Delta(\varphi|Du|^2) = (\Delta\varphi)|Du|^2 + 4\sum_{i,j=1}^{n} D_i\varphi D_juD_{ij}u + 2\varphi\sum_{i,j=1}^{n}(D_{ij}u)^2.$$

We establish the gradient estimates using a cutoff function. By taking $\varphi = \eta^2$ for some $\eta \in C_0^1(U)$ with $\eta \equiv 1$ within V, we obtain by Hölder's inequality,

$$\begin{aligned}\Delta(\eta^2|Du|^2) &= 2\eta\Delta\eta|Du|^2 + 2|D\eta|^2|Du|^2 + 8\eta\sum_{i,j=1}^{n} D_i\eta D_juD_{ij}u + 2\eta^2\sum_{ij=1}^{n}(D_{ij}u)^2 \\ &\geq (2\eta\Delta\eta - 6|D\eta|^2)|Du|^2 \geq -C|Du|^2 = -\frac{C}{2}\Delta(u^2),\end{aligned}$$

where C is a positive constant depending only on η. In the last line, we used the fact that $\Delta(u^2) = 2|Du|^2 + 2u\Delta u = 2|Du|^2$ since u is harmonic. By choosing $a \geq C/2$ large enough, we obtain

$$\Delta(\eta^2|Du|^2 + au^2) \geq 0.$$

By part (ii) of the weak maximum principle, we obtain

$$\sup_V |Du|^2 \leq \sup_V \left\{\eta^2|Du|^2 + a|u|^2\right\} \leq \sup_{\bar{U}} \left\{\eta^2|Du|^2 + a|u|^2\right\} = a\sup_{\partial U} |u|^2.$$

□

Theorem 1.5 (Removable Discontinuity)**.** *Let u be a harmonic function in $B_R(0)\backslash\{0\}$ that satisfies $u(x) = o(|x|^{2-n})$ as $|x| \longrightarrow 0$ if $n \geq 3$ or $u(x) = o(\log|x|)$ as $|x| \longrightarrow 0$ if $n = 2$. Then u can be defined at 0 so that it is smooth and harmonic in $B_R(0)$.*

Proof. For simplicity, let us only consider the case $n \geq 3$, since the case when $n = 2$ is treated exactly the same except that the fundamental solution is of the logarithmic type.

Assume u is continuous in the punctured disk $B_R(0)\backslash\{0\}$ and let v solve

$$\begin{cases} \Delta v = 0 & \text{in } B_R(0), \\ v = u & \text{on } \partial B_R(0). \end{cases}$$

Moreover, assume that $\lim_{|x|\to 0} u(x)|x|^{n-2} = 0$, i.e., any possible singularity of u at the origin grows no faster than the fundamental solution $|x|^{2-n}$ (of course, this property is trivial whenever u is bounded).

It suffices to prove that $u \equiv v$ in $B_R(0)\backslash\{0\}$. Set $w = v - u$ in $B_R(0)\backslash\{0\}$, $0 < r < R$, and $M_r := \max_{\partial B_r(0)} |w|$. Clearly,

$$|w(x)| \le M_r \frac{r^{n-2}}{|x|^{n-2}} \quad \text{on} \quad \partial B_r(0).$$

Note that both w and $\frac{1}{|x|^{n-2}}$ are harmonic in $B_R(0)\backslash B_r(0)$. Hence, the weak maximum principle implies

$$|w(x)| \le M_r \frac{r^{n-2}}{|x|^{n-2}} \quad \text{for any} \quad x \in B_R(0)\backslash B_r(0).$$

Then for each fixed $x \neq 0$,

$$|w(x)| \le \max_{\partial B_R(0)} |u| \cdot \frac{r^{n-2}}{|x|^{n-2}} + \underbrace{\frac{\max_{\partial B_r(0)} |u|}{|x|^{n-2}} \cdot r^{n-2}}_{|x|^{2-n} o(1) \text{ as } r \to 0} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 0,$$

where we used the estimate

$$M_r = \max_{\partial B_r(0)} |v-u| \le \max_{\partial B_r(0)} |v| + \max_{\partial B_r(0)} |u| \le \max_{\partial B_R(0)} |v| + \max_{\partial B_r(0)} |u| \le \max_{\partial B_R(0)} |u| + \max_{\partial B_r(0)} |u|.$$

Hence, $w \equiv 0$ in $B_R(0)\backslash\{0\}$. □

1.1.3 Further Properties of Harmonic Functions

Theorem 1.6 (Regularity). *If $u \in C(U)$ satisfies the mean value property in U, then $u \in C^\infty(U)$.*

Proof. Define $\eta \in C_c^\infty(\mathbb{R}^n)$ to be the standard mollifier

$$\eta(x) := \begin{cases} C \exp\left(\dfrac{1}{|x|^2 - 1}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \ge 1, \end{cases}$$

where $C > 0$ is chosen so that $\|\eta\|_{L^1(\mathbb{R}^n)} = 1$, and set $u_\epsilon := \eta_\epsilon * u$ in $U_\epsilon = \{x \in U \,|\, dist(x, \partial U) > \epsilon\}$. Then $u_\epsilon \in C^\infty(U_\epsilon)$. Now, the mean-value property and simple calculations imply

$$\begin{aligned} u_\epsilon(x) &= \int_U \eta_\epsilon(x-y) u(y)\, dy = \frac{1}{\epsilon^n} \int_{B_\epsilon(x)} \eta\left(\frac{|x-y|}{\epsilon}\right) u(y)\, dy \\ &= \frac{1}{\epsilon^n} \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) \left(\int_{\partial B_r(x)} u(y)\, d\sigma_y\right) dr = \frac{1}{\epsilon^n} \int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right) \frac{\omega_n}{n} r^{n-1} u(x)\, dr \\ &= u(x) \int_{B_\epsilon(0)} \eta_\epsilon(y)\, dy = u(x). \end{aligned}$$

Thus, $u \equiv u_\epsilon$ in U_ϵ and so $u \in C^\infty(U_\epsilon)$ for each $\epsilon > 0$. □

Remark 1.6. *We mention some other regularizing properties of the mollifier introduced above. If $u \in C(U)$, then $u_\epsilon \longrightarrow u$ uniformly on compact subsets of U as $\epsilon \longrightarrow 0$. Moreover, if $1 \leq p < \infty$ and the function $u \in L^p_{loc}(U)$, then $u_\epsilon \longrightarrow u$ in $L^p_{loc}(U)$.*

Theorem 1.7 (Pointwise Estimates for Derivatives)**.** *Suppose u is harmonic in U. Then*

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B_r(x))}, \tag{1.6}$$

for each ball $B_r(x) \subset U$ and each multi-index α of order $|\alpha| = k$. Particularly,

$$C_0 = \frac{n}{\omega_n}, \; C_k = \frac{(2^{n+1}k)^k n^{k+1}}{\omega_n} \; (k = 1, 2, \ldots). \tag{1.7}$$

Proof. We proceed by induction in which the case when $k = 0$ is clear. For $k = 1$, we note that derivatives of harmonic functions are also harmonic. Consequently,

$$|u_{x_i}(x)| = \Big| \frac{1}{|B_{r/2}(x)|} \int_{B_{r/2}(x)} u_{x_i}(y)\, dy \Big| = \Big| \frac{n 2^n}{\omega_n r^n} \int_{\partial B_{r/2}(x)} u(y) \nu_i \, d\sigma_y \Big| \leq \frac{2n}{r} \|u\|_{L^\infty(\partial B_{r/2}(x))} \tag{1.8}$$

If $y \in \partial B_{r/2}(x)$, then $B_{r/2}(y) \subset B_r(x) \subset U$, and so

$$|u(y)| \leq \frac{n}{\omega_n} \left(\frac{2}{r} \right)^n \|u\|_{L^1(B_r(x))},$$

where we used the estimate for the previous case $k = 0$. Inserting this into estimate (1.8) completes the verification for the case $k = 1$. Now assume that $k \geq 2$ and the estimates (1.6)–(1.7) hold for all balls in U and for each multi-index of order less than or equal to $k - 1$. Fix $B_r(x) \subset U$ and let α be a multi-index with $|\alpha| = k$. Then $D^\alpha u = (D^\beta u)_{x_i}$ for some $i \in \{1, 2, \ldots, n\}$, $|\beta| = k - 1$. Using similar calculations as before, we obtain

$$|D^\alpha u(x)| \leq \frac{nk}{r} \|D^\beta u\|_{L^\infty(\partial B_{r/k}(x))}.$$

If $y \in B_{r/k}(x)$, then $B_{\frac{k-1}{k} r}(y) \subset B_r(x) \subset U$. Thus, estimates (1.6)–(1.7) imply

$$|D^\beta u(y)| \leq \frac{n(2^{n+1} n(k-1))^{k-1}}{\omega_n (\frac{k-1}{k} r)^{n+k-1}} \|u\|_{L^1(B_r(x))}.$$

Combining the last two estimates imply the desired estimate

$$|D^\alpha u(x)| \leq \frac{n(2^{n+1} nk)^k}{\omega_n r^{n+k}} \|u\|_{L^1(B_r(x))} = \frac{C_k}{r^{n+k}} \|u\|_{L^1(B_r(x))}.$$

□

Theorem 1.8 (Liouville)**.** *Suppose $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is harmonic and bounded. Then u is constant.*

Proof. Fix $x \in \mathbb{R}^n$, $r > 0$, and apply Theorem 1.7 on $B_r(x)$ to get

$$|Du(x)| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B_r(x))} \leq \frac{C_1}{r^{n+1}} \frac{\omega_n}{n} r^n \|u\|_{L^\infty(B_r(x))} \leq \frac{C}{r} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty.$$

Hence, $Du \equiv 0$, and so u is constant. □

Theorem 1.9 (Harnack's Inequality)**.** *For each connected open set $V \subset\subset U$, there exists a positive constant $C = C(V)$, depending only on V, such that*

$$\sup_V u \leq C \inf_V u$$

for all non-negative harmonic functions u in U. In particular,

$$C^{-1} u(y) \leq u(x) \leq C u(y)$$

for all $x, y \in V$.

Remark 1.7. *Harnack's inequality asserts that non-negative harmonic functions within V are in a sense all comparable and shows that the oscillation of such functions can be controlled. Basically, a harmonic function cannot be small (large, respectively) at some point in V unless it is small (large,respectively) on all other points in V.*

Proof. Let $r := \frac{1}{4} dist(V, \partial U)$ and choose $x, y \in V$ with $|x - y| \leq r$. Then

$$\begin{aligned} u(x) &= \frac{1}{|B_{2r}(x)|} \int_{B_{2r}(x)} u(z)\, dz \geq \frac{n}{\omega_n 2^n r^n} \int_{B_r(y)} u(z)\, dz \\ &= \frac{1}{2^n} \frac{1}{|B_r(y)|} \int_{B_r(y)} u(z)\, dz = \frac{1}{2^n} u(y). \end{aligned}$$

Hence, $\frac{1}{2^n} u(y) \leq u(x) \leq 2^n u(y)$ if $x, y \in V$ with $|x - y| \leq r$. Since V is connected and its closure is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius $r/2$ and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, 3, \ldots N$. Then

$$u(x) \geq \frac{1}{2^{n(N+1)}} u(y)$$

for all $x, y \in V$. □

The following provides an another equivalent characterization of harmonic functions, and it gives a proper motivation for the notion of viscosity solutions to fully nonlinear elliptic equations (see Chapter 4).

Theorem 1.10. *Let U be a open bounded domain in $\mathbb{R}^n$. Then, u is a harmonic function in U if and only if u is continuous and satisfies the following two conditions.*

(i) If $u - \varphi$ has a local maximum at $x_0 \in U$ and $\varphi \in C^2(U)$, then $-\Delta\varphi(x_0) \leq 0$.

(ii) If $u - \varphi$ has a local minimum at $x_0 \in U$ and $\varphi \in C^2(U)$, then $-\Delta\varphi(x_0) \geq 0$.

Proof. If u is harmonic in U, then u is clearly continuous and showing it satisfies the two conditions is obvious. For instance, if $u - \varphi$ has a local maximum at $x_0 \in U$, then

$$-\Delta\varphi(x_0) = \Delta(u(x_0) - \varphi(x_0)) \leq 0.$$

The second condition is verified in a similar manner. Now suppose that the two conditions are satisfied. By regularity properties of harmonic functions as indicated earlier, we may assume that u is C^2. Then, it is clear that if $u \in C^2(U)$, then we can set $\varphi = u$ in the two conditions and conclude that $u = \varphi$ is harmonic in U. □

1.1.4 Energy and Comparison Methods for Harmonic Functions

The following are simple approaches for harmonic functions that we will make use of in the later chapters. We begin with Cacciopolli's inequality, which is sometimes called the reversed Poincaré inequality.

Lemma 1.2 (Cacciopolli's Inequality)**.** *Suppose $u \in C^1(B_1)$ satisfies*

$$\int_{B_1} a^{ij}(x) D_i u D_j \varphi \, dx = 0 \quad \textit{and} \quad \varphi \in C_0^1(B_1).$$

Then for any function $\eta \in C_0^1(B_1)$, we have

$$\int_{B_1} \eta^2 |Du|^2 \, dx \leq C \int_{B_1} |D\eta|^2 u^2 \, dx,$$

where $C = C(\lambda, \Lambda)$ is a positive constant.

Proof. For any $\eta \in C_0^1(B_1)$ set $\varphi = \eta^2 u$. From the definition of a weak solution, we have

$$\lambda \int_{B_1} \eta^2 |Du|^2 \, dx \leq \Lambda \int_{B_1} \eta |u| |D\eta| |Du| \, dx.$$

Then by Hölder's inequality,

$$\begin{aligned} \lambda \int_{B_1} \eta^2 |Du|^2 \, dx &\leq \Lambda \int_{B_1} \eta |u| |D\eta| |Du| \, dx \\ &\leq \Lambda \left(\int_{B_1} \eta^2 |Du|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_1} |D\eta|^2 u^2 \, dx \right)^{\frac{1}{2}} \end{aligned}$$

and the result follows immediately. □

Corollary 1.2. *Let u be as in Lemma 1.2. Then for any $0 \leq r < R \leq 1$, there holds*

$$\int_{B_r} |Du|^2\,dx \leq \frac{C}{(R-r)^2}\int_{B_R} |u|^2\,dx,$$

where $C = C(\lambda, \Lambda)$.

Proof. Choose η such that $\eta \equiv 1$ on B_r, $\eta \equiv 0$ outside B_R and $|D\eta| \leq 2(R-r)^{-1}$ then apply Lemma 1.2. □

Corollary 1.3. *Let u be as in Lemma 1.2. Then for any $0 < R \leq 1$, there hold*

$$\int_{B_{R/2}} u^2\,dx \leq \theta \int_{B_R} u^2\,dx, \quad \textit{and} \quad \int_{B_{R/2}} |Du|^2\,dx \leq \theta \int_{B_R} |Du|^2\,dx,$$

where $\theta = \theta(n, \lambda, \Lambda) \in (0,1)$.

Proof. Take $\eta \in C_0^1(B_R)$ with $\eta \equiv 1$ on $B_{R/2}$ and $|D\eta| \leq 2R^{-1}$. Then by Lemma 1.2 and since $D\eta \equiv 0$ in $B_{R/2}$, we have

$$\int_{B_R} |D(\eta u)|^2\,dx \leq \int_{B_R} |D\eta|^2 u^2 + \eta^2|Du|^2\,dx \leq C\int_{B_R} |D\eta|^2 u^2\,dx \leq \frac{C}{R^2}\int_{B_R\setminus B_{R/2}} u^2\,dx.$$

From this estimate and Poincaré's inequality, we obtain

$$\int_{B_{R/2}} u^2\,dx \leq \int_{B_R} (\eta u)^2\,dx \leq C_n R^2 \int_{B_R} |D(\eta u)|^2\,dx \leq C\int_{B_R\setminus B_{R/2}} u^2\,dx.$$

This further implies

$$(C+1)\int_{B_{R/2}} u^2\,dx \leq C\int_{B_R} u^2\,dx,$$

which completes the proof of the first estimate. The proof of the second estimate follows similar arguments. □

Remark 1.8. *Interestingly, Corollary 1.3 implies that every harmonic function in $\mathbb{R}^n$ with finite L^2-norm are identically zero and every harmonic function in $\mathbb{R}^n$ with finite Dirichlet integral is constant. Moreover, iterating the estimates in Corollary 1.3 leads to the following estimates. Let u be as in Lemma 1.2, then for any $0 < \rho < r \leq 1$ there hold*

$$\int_{B_\rho} u^2\,dx \leq C\left(\frac{\rho}{r}\right)^\mu \int_{B_r} u^2\,dx, \quad \textit{and} \quad \int_{B_\rho} |Du|^2\,dx \leq C\left(\frac{\rho}{r}\right)^\mu \int_{B_r} |Du|^2\,dx,$$

for some positive constant $\mu = \mu(n, \lambda, \Lambda)$. Later on we prove that we can take $\mu \in (n-2, n)$.

Lemma 1.3 (Basic Estimates for Harmonic Functions)**.** *Suppose* $\{a^{ij}\}$ *is a constant positive definite matrix satisfying the uniformly elliptic condition,*

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any} \quad \xi \in \mathbb{R}^n \tag{1.9}$$

for some $0 < \lambda \leq \Lambda$. *Suppose* $u \in C^1(B_1)$ *satisfies*

$$\int_{B_1} a^{ij}(x) D_i u D_j \varphi = 0 \quad \text{for any} \quad \varphi \in C_0^1(B_1).$$

Then for any $0 < \rho \leq r$, *there hold*

$$\int_{B_\rho} |u|^2\, dx \leq C\left(\frac{\rho}{r}\right)^n \int_{B_r} |u|^2\, dx,$$

$$\int_{B_\rho} |u - (u)_{0,\rho}|^2\, dx \leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |u - (u)_{0,r}|^2\, dx,$$

where $C = C(\lambda, \Lambda)$.

Proof. By dilation, consider $r = 1$. We restrict our consideration to the range $\rho \in (0, 1/2]$, since the estimates are trivial for when $\rho \in (1/2, 1]$.
Claim:

$$\|u\|^2_{L^\infty(B_{1/2})} + \|Du\|^2_{L^\infty(B_{1/2})} \leq C(\lambda, \Lambda) \int_{B_1} |u|^2\, dx.$$

From this we get

$$\int_{B_\rho} |u|^2\, dx \leq \rho^n \|u\|^2_{L^\infty(B_{1/2})} \leq c\rho^n \int_{B_1} |u|^2\, dx$$

and

$$\int_{B_\rho} |u - u_\rho|^2\, dx \leq \int_{B_\rho} |u - u(0)|^2\, dx \leq \rho^{n+2} \|Du\|^2_{L^\infty(B_{1/2})} \leq c\rho^{n+2} \int_{B_1} |u|^2\, dx.$$

If u is a solution of (1.9) then so is $u - u_1$. With u replaced by $u - u_1$ in the above inequality, there holds

$$\int_{B_\rho} |u - u_\rho|^2\, dx \leq c\rho^{n+2} \int_{B_1} |u - u_1|^2\, dx.$$

It remains to prove the claim. If u is a solution of (1.9), then so are any derivatives of u. By applying Corollary 1.2 to the derivatives of u, we conclude that for any positive integer k

$$\|u\|_{H^k(B_{1/2})} \leq c(k, \lambda, \Lambda)\|u\|_{L^2(B_1)}.$$

By fixing k sufficiently large, the Sobolev embedding theorem implies that $H^k(B_{1/2}) \hookrightarrow C^1(\bar{B}_{1/2})$. Thus,

$$\|u\|_{C^1(\bar{B}_{1/2})} = \sup_{\bar{B}_{1/2}} |u(x)| + \sup_{\bar{B}_{1/2}} |Du(x)| \leq c(n)\|u\|_{H^k(B_{1/2})} \leq c(n, k, \lambda, \Lambda)\|u\|_{L^2(B_1)}.$$

This completes the proof of the lemma. $\square$

1.2 Classical Maximum Principles

In this section, we consider an elliptic operator L in non-divergence form:

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

where the coefficients a^{ij}, b^i, c are continuous in some bounded open subset $U \subset \mathbb{R}^n$ and the uniform ellipticity condition holds. We now introduce the important maximum principles for second-order uniformly elliptic equations. In the next chapter, we will instead focus on uniformly elliptic operators in divergence form, which are more appropriate for the energy and variational methods introduced in that chapter. In the later chapters, we will also look at maximum principles for weak solutions when we study the weak Harnack inequality and its connection with regularity properties of solutions to elliptic equations (see Theorem 3.33 for example).

1.2.1 The Weak Maximum Principle

Theorem 1.11 (Weak Maximum Principle). *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$ in U.*

(a) If $Lu \leq 0$ in U, then $\max_{\bar{U}} u = \max_{\partial U} u$.

(b) If $Lu \geq 0$ in U, then $\min_{\bar{U}} u = \min_{\partial U} u$.

Proof. We prove assertion (a).

Step 1: First we assume $Lu < 0$ in U but there exists $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$. Of course, at this maximum point there hold

$$(i)\ Du(x_0) = 0 \ \text{ and } \ (ii)\ D^2u(x_0) \leq 0. \tag{1.10}$$

Since $A = (a^{ij}(x_0))$ is symmetric and positive definite, there is an orthogonal matrix $O = (o_{ij})$ such that

$$OAO^T = diag(d_1, d_2, \ldots, d_n), \tag{1.11}$$

where $OO^T = I$ and $d_k > 0$ for $k = 1, 2, \ldots, n$. Write $y = x_0 + O(x - x_0)$ so that $x - x_0 = O^T(y - x_0)$,

$$u_{x_i} = \sum_{k=1}^{n} u_{y_k} o_{ki} \ \text{ and } \ u_{x_ix_j} = \sum_{k,\ell=1}^{n} u_{y_ky_\ell} o_{ki} o_{\ell j} \ (i, j = 1, 2, \ldots, n).$$

Hence, at the point x_0,

$$\begin{aligned}\sum_{i,j=1}^{n} a^{ij} u_{x_ix_j} &= \sum_{k,\ell=1}^{n} \sum_{k,\ell=1}^{n} a^{ij} u_{y_ky_\ell} o_{ki} o_{\ell j} \\ &= \sum_{k=1}^{n} d_k u_{y_ky_k} \leq 0, \end{aligned} \tag{1.12}$$

where in the last line the inequality is due to (1.10)(ii) and the fact that $d_k > 0$ for $k = 1, 2, \ldots, n$, and the equality is due to (1.11). From (1.10)(i) and (1.12), at the point x_0 we have

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} \geq 0,$$

and we arrive at a contradiction.

Step 2: Now we complete the proof for the case when $Lu \geq 0$ in U. Set

$$u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1},\ x \in U,$$

where $\lambda > 0$ will be specified below and $\epsilon > 0$. From the uniform ellipticity condition, there holds $a^{ii}(x) \geq \theta$ for $i = 1, 2, \ldots, n$, $x \in U$. Hence,

$$Lu^\epsilon = Lu + \epsilon L(e^{\lambda x_1}) \leq \epsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1) \leq \epsilon e^{\lambda x_1}(-\lambda^2 \theta + \|b\|_{L^\infty} \lambda) < 0 \text{ in } U,$$

provided that $\lambda > 0$ is chosen to be sufficiently large. Namely, we have $Lu^\epsilon > 0$ in U and we conclude $\max_{\bar{U}} u^\epsilon = \max_{\partial U} u^\epsilon$ from step 1. Let $\epsilon \longrightarrow 0$ to find $\max_{\bar{U}} u = \max_{\partial U} u$.

Assertion (b) follows easily from (a) once we make the simple observation that $-u$ is a subsolution, i.e., $L(-u) \leq 0$ in U whenever u is a supersolution. □

1.2.2 The Strong Maximum Principle

Just as we have for harmonic functions, the weak maximum principles may be strengthened after some added conditions on U. In order to do this, we make use of Hopf's Lemma.

Lemma 1.4 (Hopf's Lemma). *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$ in U. Suppose further that $Lu \geq 0$ in U and there is a ball B contained in U with a point $x^0 \in \partial U \cap \partial B$ such that*

$$u(x) > u(x^0) \text{ for all } x \in B. \tag{1.13}$$

(a) Then for any outward directional derivative at x^0,

$$\frac{\partial u}{\partial \nu}(x^0) < 0.$$

(b) If $c \geq 0$ in U, the same conclusion holds provided $u(x^0) \leq 0$.

Remark 1.9. *An analogous result holds for when $Lu \leq 0$ in U but with the inequalities in the above "interior ball" condition and the conclusions are switched to be in the opposite direction.*

Proof of Hopf's Lemma. Assume $c \geq 0$ and also assume, without loss of generality, that $B = B_r(0)$ for some $r > 0$.
Step 1: Define

$$v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2} \text{ for } x \in B_r(0)$$

for $\lambda > 0$ to be specified below. Then, from the uniform ellipticity condition,

$$\begin{aligned}
Lv = & -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i v_{x_i} + cv \\
&= e^{-\lambda|x|^2} \sum_{i,j=1}^{n} a^{ij}(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}) - e^{-\lambda|x|^2} \sum_{i=1}^{n} b^i 2\lambda x_i + c(e^{-\lambda|x|^2} - e^{-\lambda r^2}) \\
&\leq e^{-\lambda|x|^2}(-4\theta\lambda^2|x|^2 + 2\lambda tr(A) + 2\lambda|b||x| + c),
\end{aligned}$$

for $A = (a^{ij})$ and $b = (b^i)$. Next consider the open annulus $R = B_r^0(0) \backslash B_{r/2}(0)$ and so

$$Lv \leq e^{-\lambda|x|^2}(-\theta\lambda^2 r^2 + 2\lambda tr(A) + 2\lambda|b|r + c) \leq 0 \text{ in } R \tag{1.14}$$

provided that $\lambda > 0$ is fixed to be large enough.
Step 2: In view of (1.13), there exists a constant $\epsilon > 0$ small for which

$$u(x^0) \geq u(x) + \epsilon v(x) \text{ for } x \in \partial B_{r/2}(0). \tag{1.15}$$

In addition, notice since $v \equiv 0$ on $\partial B_r(0)$,

$$u(x^0) \geq u(x) + \epsilon v(x) \text{ for } x \in \partial B_r(0). \tag{1.16}$$

Step 3: From (1.14), we see

$$L(u + \epsilon v - u(x^0)) \leq -cu(x^0) \leq 0 \text{ in } R,$$

and from (1.15) and (1.16) we have

$$u + \epsilon v - u(x^0) \leq 0 \text{ on } \partial R.$$

The weak maximum principle implies that $u+\epsilon v-u(x^0) \leq 0$ in R, but $u(x^0)+\epsilon v(x^0)-u(x^0) = 0$, and so

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0.$$

Consequently,

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda\epsilon r e^{-\lambda r^2} > 0.$$

This completes the proof. $\square$

Theorem 1.12 (Strong Maximum Principle)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$, $c \equiv 0$ in $U \subset \mathbb{R}^n$, and U is connected, open and bounded.*

(a) If $Lu \leq 0$ in U and u attains its maximum over $\bar{U}$ at an interior point, then u is constant within U.

(b) If $Lu \geq 0$ in U and u attains its minimum over $\bar{U}$ at an interior point, then u is constant within U.

Proof. We prove statement (a) only, since statement (b) follows similarly. Write $M = \max_{\bar{U}} u$ and take $C = \{x \in U \,|\, u(x) = M\}$. If C is empty or if $u \equiv M$ we are done. Otherwise, if $u \not\equiv M$, set

$$V = \{x \in U \,|\, u(x) < M\}.$$

Choose a point $y \in V$ satisfying $dist(y, C) < dist(y, \partial U)$ and let B denote the largest ball with center y whose interior lies in V. Then there exists some point $x^0 \in C$ with $x^0 \in \partial B$. It is easy to check that V satisfies the interior ball condition at x^0. Hence, by part (a) of Hopf's lemma, $\partial u/\partial\nu(x^0) > 0$. But this contradicts with the fact that $Du(x^0) = 0$ since u attains its maximum at $x^0 \in U$. $\square$

If the coefficient $c(x)$ is non-negative, then we have the following version of the strong maximum principle. Its proof is the same as before but invokes statement (b) in Hopf's lemma.

Theorem 1.13 (Strong Maximum Principle for $c \geq 0$)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$, $c \geq 0$ in $U \subset \mathbb{R}^n$, and U is connected, open and bounded.*

(a) If $Lu \leq 0$ in U and u attains a non-negative maximum over $\bar{U}$ at an interior point, then u is constant within U.

(b) If $Lu \geq 0$ in U and u attains a non-positive minimum over $\bar{U}$ at an interior point, then u is constant within U.

Finally, we state a quantitative version of the maximum principle for second-order elliptic equations called Harnack's inequality. However, a more general version with proof shall be offered in Chapter 3. There we will see the importance of Harnack's inequality and how it applies to obtaining several results on a weaker notion of solution, called weak or distributional solutions, for elliptic equations. This includes results on their regularity properties, Liouville type theorems, and even a version of the strong maximum principle adapted to weak solutions.

Theorem 1.14. *Assume u is a non-negative C^2 solution of*

$$Lu = 0 \ \text{ in } \ U,$$

and suppose $V \subset\subset U$ is connected. Then there exists a constant C such that

$$\sup_V u \le C \inf_V u.$$

The constant C depends only on V and the coefficients of L.

1.3 Newtonian and Riesz Potentials

1.3.1 The Newtonian Potential and Green's Formula

Definition 1.2. *The function*

$$\Gamma(x) := \begin{cases} \dfrac{1}{2\pi} \log|x|, & \text{if } n = 2, \\ \dfrac{1}{\omega_n(n-2)} \dfrac{1}{|x|^{n-2}}, & \text{if } n \ge 3. \end{cases}$$

defined for all $x \in \mathbb{R}^n \backslash \{0\}$, is the fundamental solution of Laplace's equation. In addition, if $f \in L^p(U)$ for $1 < p < \infty$, then the Newtonian potential of f is defined by

$$w(x) := \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy.$$

The following theorem is a basic result which states that the kernel Γ in the Newtonian potential is the fundamental solution of Poisson's equation. We refer the reader to the references introduced earlier for a proof of this elementary result.

Theorem 1.15. *Let $f \in C_c^2(\mathbb{R}^n)$ and define u to be the Newtonian potential of f. Then*

(i) $u \in C^2(\mathbb{R}^n)$,

(ii) $-\Delta u = f$ *in* $\mathbb{R}^n$.

Proof. **Step 1:** Clearly,

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Gamma(y) f(x-y)\, dy,$$

therefore,

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Gamma(y) \Big(\frac{f(x+he_i-y) - f(x-y)}{h} \Big)\, dy,$$

where $h \neq 0$ and $e_i = (0, \ldots, 1, 0, \ldots 0)$ where the 1 is in the i^{th} slot. Of course,

$$\frac{f(x+he_i-y) - f(x-y)}{h} \longrightarrow f_{x_i}(x-y) \text{ uniformly on } \mathbb{R}^n \text{ as } h \longrightarrow 0,$$

and thus for $i = 1, 2, \ldots, n$,

$$u_{x_i}(x) = \int_{\mathbb{R}^n} \Gamma(y) f_{x_i}(x - y)\, dy.$$

Likewise, for $i = 1, 2, \ldots, n$,

$$u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Gamma(y) f_{x_i x_j}(x - y)\, dy$$

and this shows u is C^2 since the right-hand side of the last identity is continuous.

Step 2: Fix $\varepsilon > 0$ and suppose $n \geq 3$. Due to the singularity of fundamental solution at the origin, we must be careful in our calculation. Namely, we first consider the splitting

$$\Delta u(x) = \int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x - y)\, dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y) \Delta_x f(x - y)\, dy := I^1_\varepsilon + I^2_\varepsilon. \tag{1.17}$$

Then, polar coordinates implies

$$|I^1_\varepsilon| \leq C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B_\varepsilon(0)} |\Gamma(y)|\, dy \leq C\varepsilon^{n-(n-2)} \leq C\varepsilon^2. \tag{1.18}$$

Integration by parts implies

$$\begin{aligned} I^2_\varepsilon &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y) \Delta_y f(x - y)\, dy \\ &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} D\Gamma(y) \cdot D_y f(x - y)\, dy + \int_{\partial B_\varepsilon(0)} \Gamma(y) \frac{\partial f}{\partial \nu}(x - y)\, dS(y) \\ &:= J^1_\varepsilon + J^2_\varepsilon, \end{aligned} \tag{1.19}$$

where ν denotes the inward pointing unit normal along $\partial B_\varepsilon(0)$. Now,

$$|J^2_\varepsilon| \leq \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B_\varepsilon(0)} |\Gamma(y)|\, dS(y) \leq C\varepsilon. \tag{1.20}$$

Again, integration by parts and since Γ is harmonic away from the origin, we get

$$\begin{aligned} J^1_\varepsilon &= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Delta\Gamma(y) f(x - y)\, dy - \int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial \nu}(y) f(x - y)\, dS(y) \\ &= -\int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma}{\partial \nu}(y) f(x - y)\, dS(y). \end{aligned} \tag{1.21}$$

Now, it is clear that $D\Gamma(y) = -\frac{1}{\omega_n} \frac{y}{|y|^n}$ $(y \neq 0)$ and $\nu = -y/|y| = -y/\varepsilon$ on $\partial B_\varepsilon(0)$. Thus,

$$\frac{\partial \Gamma}{\partial \nu}(y) = \nu \cdot D\Gamma(y) = \frac{1}{\omega_n \varepsilon^{n-1}} \text{ on } \partial B_\varepsilon(0).$$

Hence,

$$J_\varepsilon^1 = -\frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)\, dS(y) = -\frac{1}{|B_\varepsilon(0)|} \int_{\partial B_\varepsilon(0)} f(x-y)\, dS(y) \longrightarrow -f(x) \quad (1.22)$$

as $\varepsilon \longrightarrow 0$. Hence, combining the estimates (1.18)–(1.22) and sending $\varepsilon \longrightarrow 0$ in (1.17), we obtain $-\Delta u(x) = f(x)$ and this completes the proof. □

Remark 1.10. *The proof above remains valid in the case where $n = 2$ except that the estimates for I_ε^1 and J_ε^2 become*

$$|I_\varepsilon^1| \leq C\varepsilon^2 |\log \varepsilon| \;\; \textit{and} \;\; |J_\varepsilon^2| \leq C\varepsilon |\log \varepsilon|.$$

1.3.2 Riesz Potentials and the Hardy-Littlewood-Sobolev Inequalitiy

From the previous theorem, we see that the Newtonian potential provides an explicit formula for solutions of Poisson's equation. On the other hand, the integral equation provides a simple example of a singular integral operator, which can be naturally extended to more general singular integral operators such as the Riesz potential. Remarkably yet not surprisingly, the Riesz potentials are very closely related to problems involving fractional Laplacians such as the Lane-Emden and Hardy-Littlewood-Sobolev systems. We give a definition of Riesz potentials here and briefly discuss their boundedness in L^p spaces.

Definition 1.3. *Let α be a complex number with positive real part Re $\alpha > 0$. The Riesz potential of order α is the operator*

$$I_\alpha = (-\Delta)^{-\alpha/2}.$$

In particular,

$$I_\alpha(f)(x) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

where $C_{n,\alpha} = 2^{-\alpha}\pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}$ and the integral is convergent if $f \in \mathcal{S}$, i.e., f belongs in the Schwartz class.

The following result is the well-known Hardy-Littlewood-Sobolev inequality, which shows the boundedness of the Riesz potentials. The proof of these theorems can be found in [5, 16].

Theorem 1.16 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \alpha < n$ and $p, q > 1$ such that*

$$\frac{1}{p} + \frac{1}{q} + \frac{n-\alpha}{n} = 2.$$

Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}}\, dxdy \leq C_{n,p,\alpha} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \quad (1.23)$$

for any $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ where $C_{n,p,\alpha}$ is a positive constant.

Remark 1.11. *The sharp constant in the HLS inequality satisfies*

$$C_{n,\lambda,p} \leq \frac{n}{(n-\lambda)pq} \left(\frac{|\mathbb{S}^{n-1}|}{n}\right)^{\lambda/n} \left(\left(\frac{\lambda/n}{1-1/p}\right)^{\lambda/n} + \left(\frac{\lambda/n}{1-1/q}\right)^{\lambda/n}\right),$$

where $\lambda = n - \alpha$.

The following is an equivalent formulation of the Hardy-Littlewood-Sobolev inequality. It determines the conditions on the exponents p and q that guarantee $I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$ is a bounded linear operator. For completeness, we shall give a proof of this version of the Hardy-Littlewood-Sobolev inequality in Section 3.1.5.

Theorem 1.17. *Let* $\alpha \in (0, n)$, $1 < p \leq q < \infty$, $f \in L^p(\mathbb{R}^n)$ *and*

$$\frac{n}{n-\alpha} < q \ \ \textit{with} \ \ \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}, \ \textit{i.e.,} \ \ p = \frac{np}{n+\alpha p}.$$

Then

$$\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leq C_{n,p,\alpha} \|f\|_{L^p(\mathbb{R}^n)}.$$

One interesting motivation for considering Riesz potentials is due to their close relationship with poly-harmonic equations. For instance, consider the system

$$\begin{cases} (-\Delta)^{\alpha/2} u = |x|^{\sigma_1} v^q, \quad u > 0, \quad \text{in } \mathbb{R}^n, \\ (-\Delta)^{\alpha/2} v = |x|^{\sigma_2} u^p, \quad v > 0, \quad \text{in } \mathbb{R}^n. \end{cases} \tag{1.24}$$

When $\alpha \in (0, n)$ is an even integer and $\sigma_1, \sigma_2 \in (-\alpha, \infty)$, (1.24) is equivalent to the integral system of Riesz potentials

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|y|^{\sigma_1} v(y)^q}{|x-y|^{n-\alpha}} dy, \ u > 0 \ \text{ in } \mathbb{R}^n, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{|y|^{\sigma_2} u(y)^p}{|x-y|^{n-\alpha}} dy, \ v > 0 \ \text{ in } \mathbb{R}^n, \end{cases} \tag{1.25}$$

in the sense that a classical solution of one system, multiplied by a suitable constant if necessary, is also a solution of the other when $p, q > 1$, and vice versa. Interestingly, when $\sigma_i = 0$, the integral equations in (1.25) are the Euler–Lagrange equations of a functional under a constraint in the context of the HLS inequality. In particular, the extremal functions for obtaining the sharp constant in the HLS inequality are solutions of the system of integral equations. For more on the analysis of systems (1.24) and (1.25), we refer the reader to the papers [14, 15, 23, 24, 25, 26] and the references therein.

1.3.3 Green's Function and Representation Formulas of Solutions

Let $U \subset \mathbb{R}^n$ be an open and bounded subset with C^1 boundary ∂U. Our goal here is to find a representation of the solution of Poisson's equation

$$-\Delta u = f \text{ in } U$$

subject to the prescribed boundary condition

$$u = g \text{ on } \partial U.$$

We derive the formula for the Green's function to this problem. Fix $x \in U$ and choose $\epsilon > 0$ suitably small so that $B_\epsilon(x) \subset U$. Then, apply Green's formula on the region $V_\epsilon = U \backslash B_\epsilon(x)$ to $u(y)$ and $\Gamma(y-x)$ to get

$$\int_{V_\epsilon} u(y)\Delta\Gamma(y-x) - \Gamma(y-x)\Delta u(y)\,dy = \int_{\partial V_\epsilon} u(y)\frac{\partial \Gamma}{\partial \nu}(y-x) - \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y). \quad (1.26)$$

Notice that $\Delta\Gamma(x-y) = 0$ for $x \neq y$ and that

$$\Big| \int_{\partial B_\epsilon(x)} \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y)\Big| \leq C\epsilon^{n-1} \max_{\partial B_\epsilon(0)} |\Gamma| = o(1).$$

Then, similar to the proof of Theorem 1.15, we can show that

$$\int_{\partial B_\epsilon(x)} u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\,dS(y) = \frac{1}{|\partial B_\epsilon(x)|}\int_{\partial B_\epsilon(x)} u(y)\,dS(y) \longrightarrow u(x)$$

as $\epsilon \longrightarrow 0$. Hence, sending $\epsilon \longrightarrow 0$ in (1.26) yields

$$u(x) = \int_{\partial U} \left\{\Gamma(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu}(y-x)\right\} dS(y) - \int_U \Gamma(y-x)\Delta u(y)\,dy. \quad (1.27)$$

Indeed, identity (1.27) holds for any point $x \in U$ and any function $u \in C^2(U)$. This representation of u is almost complete since we know u satisfies Poisson's equation and its values on the boundary are given, i.e., we know the values of Δu in U and $u = g$ on ∂U. However, we do not know a priori the value of $\partial u/\partial \nu$ on ∂U. To circumvent this, we introduce, for fixed $x \in U$, a corrector function $\phi^x = \phi^x(y)$, solving the boundary-value problem

$$\begin{cases} \Delta\phi^x = 0 & \text{in } U, \\ \phi^x = \Gamma(y-x) & \text{on } \partial U. \end{cases} \quad (1.28)$$

As before, if we apply Green's formula once more, we obtain

$$\begin{aligned} -\int_U \phi^x(y)\Delta u(y)\,dy &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \phi^x(y)\frac{\partial u}{\partial \nu}(y)\,dS(y) \\ &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \Gamma(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y). \quad (1.29) \end{aligned}$$

Now introduce the Green's function for the region U.

Definition 1.4. *The Green's function for the region U is*

$$G(x,y) := \Gamma(y-x) - \phi^x(y) \ \text{ for } \ x,y \in U, \ x \neq y.$$

In view of this definition, adding (1.29) to (1.27) yields

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)\, dS(y) - \int_U G(x,y)\Delta u(y)\, dy \quad (x \in U), \tag{1.30}$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = D_y G(x,y) \cdot \nu(y)$$

is the outer normal derivative of G with respect to the variable y. Here, observe that the term $\partial u/\partial \nu$ no longer appears in identity (1.30).

In summary, suppose that $u \in C^2(\bar{U})$ is a solution of the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U, \\ \quad\;\; u = g & \text{on } \partial U, \end{cases} \tag{1.31}$$

for given continuous functions f and g. Then, we have basically shown the following.

Theorem 1.18 (Representation formula via Green's function)**.** *If $u \in C^2(\bar{U})$ solves problem* (1.31)*, then*

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\, dS(y) + \int_U G(x,y) f(y)\, dy \quad (x \in U). \tag{1.32}$$

If the geometry of U is simple enough, then we can actually compute the corrector function explicitly to obtain G. Two such examples are when U is the unit ball or the hyperbolic or half-space in $\mathbb{R}^n$.

1.3.4 Green's Function for a Half-Space

Consider the half-space

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,|\, x_n > 0\},$$

whose boundary is given by $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. Although the half-space is unbounded and the calculations in the previous section assumed U was bounded, we can still use the same ideas to find the Green's function for the half-space. In order to do so, we adopt a reflection argument. Namely, if $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, we let $\tilde{x} = (x_1, x_2, \ldots, -x_n)$, the reflection of x in the plane $\partial\mathbb{R}^n_+$. Then set

$$\phi^x(y) = \Gamma(y - \tilde{x}) = \Gamma(y_1 - x_1, \ldots, y_{n-1} - x_{n-1}, y_n + x_n) \ \text{ for } \ x, y \in \mathbb{R}^n_+.$$

The idea is that this corrector ϕ^x is built from Γ by reflecting the singularity from $x \in \mathbb{R}^n_+$ to $\tilde{x} \notin \mathbb{R}^n_+$. Observe that

$$\phi^x(y) = \Gamma(y-x) \text{ if } y \in \partial\mathbb{R}^n_+,$$

and thus

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+, \\ \quad \phi^x = \Gamma(y-x) & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.33}$$

as required. That is, we have the following definition.

Definition 1.5. *The Green's function for the half-space $\mathbb{R}^n_+$ is*

$$G(x,y) := \Gamma(y-x) - \Gamma(y-\tilde{x}) \text{ for } x, y \in \mathbb{R}^n_+,\ x \neq y.$$

Then

$$G_{y_n}(x,y) = \Gamma_{y_n}(y-x) - \Gamma_{y_n}(y-\tilde{x}) = \frac{-1}{\omega_n}\Big[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n}\Big].$$

Consequently, if $y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G}{\partial \nu}(x,y) = -G_{y_n}(x,y) = -\frac{2x_n}{\omega_n}\frac{1}{|x-y|^n}.$$

Now if u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ \quad u = g & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.34}$$

then the representation formula (1.32) of the previous theorem suggests that

$$u(x) = \frac{2x_n}{\omega_n}\int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy \quad (x \in \mathbb{R}^n_+) \tag{1.35}$$

is the representation formula for the solution. Here, the function

$$K(x,y) := \frac{2x_n}{\omega_n}\frac{1}{|x-y|^n} \text{ for } x \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+$$

is called **Poisson's kernel** for $U = \mathbb{R}^n_+$ and (1.35) is called **Poisson's formula**. Now, let us prove that Poisson's formula indeed gives the formula for the solution of the boundary-value problem (1.34).

Theorem 1.19 (Poisson's formula for $\mathbb{R}^n_+$)**.** *Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, and define u by Poisson's formula* (1.35)*. Then*

(a) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(b) $\Delta u = 0$ *in* $\mathbb{R}^n_+$,

(c) $\displaystyle\lim_{x\to x^0, x\in\mathbb{R}^n_+} u(x) = g(x^0)$ *for each point* $x^0 \in \partial\mathbb{R}^n_+$.

1.3.5 Green's Function for a Ball

If $U = B_1(0)$, we construct the Green's function through another reflection argument, but here we exploit an inversion through the unit sphere $\partial B_1(0)$.

Definition 1.6. *If $x \in \mathbb{R}^n \backslash \{0\}$, the point*

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the point dual to x with respect to $\partial B_1(0)$. The mapping $x \mapsto \tilde{x}$ is inversion through the unit sphere $\partial B_1(0)$.

Obviously, the inversion maps points on the sphere to itself, maps the points in the ball to its exterior $\mathbb{R}^n \backslash B_1(0)$, and maps points in the exterior into the ball. Now fix $x \in B_1(0)$ and we want to find the corrector function $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta \phi^x = 0 & \text{in } B_1(0), \\ \phi^x = \Gamma(y - x) & \text{on } \partial B_1(0), \end{cases} \tag{1.36}$$

with the Green's function

$$G(x, y) = \Gamma(y - x) - \phi^x(y).$$

Notice that the mapping $y \mapsto \Gamma(y - \tilde{x})$ is harmonic for $y \neq \tilde{x}$. Thus $y \mapsto |x|^{2-n}\Gamma(y - \tilde{x})$ is harmonic for $y \neq \tilde{x}$. Hence,

$$\phi^x(y) := \Gamma(|x|(y - \tilde{x})) \tag{1.37}$$

is harmonic in $U = B_1(0)$. Furthermore, if $y \in \partial B_1(0)$ and $x \neq 0$,

$$|x|^2|y - \tilde{x}|^2 = |x|^2\Big(|y|^2 - 2\frac{y \cdot x}{|x|^2} + \frac{1}{|x|^2}\Big) = |x|^2 - 2y \cdot x + 1 = |x - y|^2.$$

That is, $|x - y|^{2-n} = (|x||y - \tilde{x}|)^{2-n}$ and so

$$\phi^x(y) = \Gamma(y - x) \quad (y \in \partial B_1(0)),$$

as required.

Definition 1.7. *The Green's function for the unit ball $B_1(0)$ is*

$$G(x, y) := \Gamma(y - x) - \Gamma(|x|(y - \tilde{x})) \quad (x, y \in B_1(0)). \tag{1.38}$$

Note that the same formula holds when $n = 2$, where the kernel Γ is of the logarithmic type. Now assume u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B_1(0), \\ u = g & \text{on } \partial B_1(0). \end{cases} \tag{1.39}$$

Then the representation formula (1.32) indicates that

$$u(x) = -\int_{\partial B_1(0)} g(y) \frac{\partial G}{\partial \nu}(x, y)\, dS(y). \tag{1.40}$$

Then, according to (1.38),

$$G_{y_i}(x, y) = \Gamma_{y_i}(y - x) - \Gamma(|x|(y - \tilde{x}))_{y_i}.$$

We calculate that

$$\Gamma_{y_i}(y - x) = \frac{1}{\omega_n} \frac{x_i - y_i}{|x - y|^n},$$

and

$$\Gamma(|x|(y - \tilde{x}))_{y_i} = -\frac{1}{\omega_n} \frac{y_i |x|^2 - x_i}{(|x||y - \tilde{x}|)^n} = -\frac{1}{\omega_n} \frac{y_i |x|^2 - x_i}{|x - y|^n}$$

if $y \in \partial B_1(0)$. Then,

$$\frac{\partial G}{\partial \nu}(x, y) = \sum_{i=1}^{n} y_i G_{y_i}(x, y) = -\frac{1}{\omega_n} \frac{1}{|x - y|^n} \sum_{i=1}^{n} y_i((y_i - x_i) - y_i |x|^2 + x_i) = -\frac{1}{\omega_n} \frac{1 - |x|^2}{|x - y|^n}.$$

Inserting this into (1.40) yields the representation formula

$$u(x) = \frac{1 - |x|^2}{\omega_n} \int_{\partial B_1(0)} \frac{g(y)}{|x - y|^n}\, dS(y).$$

Actually, we can use a dilation argument to get the Green's function for $U = B_R(0)$. Namely, suppose now that u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } B_R(0), \\ u = g & \text{on } \partial B_R(0). \end{cases} \tag{1.41}$$

It is easy to check that $\tilde{u}(x) = u(Rx)$ solves (1.39) with $\tilde{g} = g(Rx)$ replacing g. A simple change of variables yields **Poisson's formula**

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(y)}{|x - y|^n}\, dS(y) \quad (x \in B_R(0)), \tag{1.42}$$

where the function

$$K(x, y) := \frac{R^2 - |x|^2}{\omega_n R} \frac{1}{|x - y|^n} \quad (x \in B_R(0), y \in \partial B_R(0))$$

is **Poisson's kernel** for the ball $U = B_R(0)$.

We have established Poisson's formula (1.43) under the assumption that a smooth solution of (1.41) exists. Indeed, the following theorem asserts that this formula does indeed give a solution.

Theorem 1.20 (Poisson's formula for the ball $B_R(0)$)**.** *Assume $g \in C(\partial B_R(0))$ and define u by Poisson's formula* (1.43)*. Then*

(a) $u \in C^\infty(B_R(0))$,

(b) $\Delta u = 0$ *in* $B_R(0)$,

(c) $\displaystyle\lim_{x \to x^0, x \in B_R(0)} u(x) = g(x^0)$ *for each point* $x^0 \in \partial B_R(0)$.

Observe that Harnack's inequality can be established directly from Poisson's formula (1.43).

Theorem 1.21 (Harnack's inequality)**.** *Suppose u is a non-negative harmonic function in $B_R(x_0)$. Then*

$$\left(\frac{R}{R+r}\right)^{n-2} \frac{R-r}{R+r} u(x_0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2} \frac{R+r}{R-r} u(x_0)$$

where $r = |x - x_0| < R$.

Proof. By the regularity and translation invariance properties of harmonic functions, we may assume $x_0 = 0$ and $u \in C(\bar{B}_R)$. Thus, from Poisson's formula,

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{u(y)}{|x-y|^n} \, dS(y) \quad (x \in B_R(0)). \tag{1.43}$$

Now, since $R - |x| \leq |x - y| \leq R + |x|$ for $|y| = R$, we obtain

$$\frac{1}{\omega_n R} \frac{R-|x|}{R+|x|} \left(\frac{1}{R+|x|}\right)^{n-2} \int_{\partial B_R} u(y) \, dS \leq u(x) \leq \frac{1}{\omega_n R} \frac{R+|x|}{R-|x|} \left(\frac{1}{R-|x|}\right)^{n-2} \int_{\partial B_R} u(y) \, dS.$$

In view of the mean value property,

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} u(y) \, dS,$$

we insert this into the previous estimates to arrive at the desired result. □

From this, we deduce the Liouville theorem.

Corollary 1.4. *If u is an entire function, i.e., it is harmonic in $U = \mathbb{R}^n$, and u is either bounded above or below, then u is necessarily constant.*

Proof. By shifting, we may assume u is non-negative in $\mathbb{R}^n$. Then take any point $x \in \mathbb{R}^n$ and apply the previous Harnack's inequality to u on any ball $B_R(0)$ with $|x| < R$ to get

$$\left(\frac{R}{R+|x|}\right)^{n-2} \frac{R-|x|}{R+|x|} u(0) \leq u(x) \leq \left(\frac{R}{R-|x|}\right)^{n-2} \frac{R+|x|}{R-|x|} u(0).$$

Sending $R \longrightarrow +\infty$ here yields $u(x) = u(0)$, and we conclude that u is constant everywhere in $\mathbb{R}^n$ since x was chosen arbitrarily. □

1.4 Hölder Regularity for Poisson's Equation

Let us motivate the consideration of Hölder spaces $C^{k,\alpha}$ rather than the classical C^k spaces when dealing with regularity and solvability of elliptic problems of the form $Lu = f$ in U.

For instance, if $f \in C_0^\infty(U)$ and $\Gamma = \Gamma(x)$ is the fundamental solution of Laplace's equation, then the Newtonian potential of f, i.e., $w = \Gamma * f$ or

$$w(x) = \int_U \Gamma(x-y) f(y)\, dy,$$

belongs to $C^\infty(\bar{U})$. However, if f is merely just continuous, then w is not necessarily twice differentiable.

Generally, $Lu = f$ in U is uniquely solvable for all $f \in C^2(U)$ in that there exists a unique solution $u \in C^2(U)$ for each such f; namely, the elliptic operator $L : C^2(U) \to C^2(U)$ is a bijective mapping. On the other hand, we naturally ask if for every $f \in C(U)$ the equation $Lu = f$ has a solution u in $C^2(U)$. Interestingly enough, this is not true and so the mapping $L : C^2(U) \to C(U)$ is not bijective. For instance, if $L = -\Delta$ or $L = -(\Delta - 1)$ and for the equation $Lu = f$, it is not true that for every $f \in C(U)$ the corresponding solution u belongs in $C^2(U)$ (see the example given below). Fortunately, if we hope to recover the bijectivity of the map L, we must instead consider the Hölder space $C^\alpha(U)$ in place of $C(U)$.

Remark 1.12. *One instance where the bijectivity (namely, the invertibility) of the map L becomes very important is in the method of continuity (see Section 2.6). This method makes use of the bijection of the solution map and the global $C^{2,\alpha}$ regularity estimates to prove existence results to general elliptic boundary value problems. Therefore, this gives further motivation and a glimpse of some topics examined in the later chapters.*

Example: Let us provide an example in which the solvability of $-\Delta u = f$ for a carefully chosen continuous f fails within the class of C^2 solutions. Take the continuous but not Hölder continuous function

$$f(x) = \frac{x_1^2 - x_2^2}{2|x|^2}\Big(\frac{n+2}{(-\log|x|)^{1/2}} + \frac{1}{2(-\log|x|)^{3/2}}\Big),$$

set

$$g(x) = \sqrt{-\log R}(x_2^2 - x_1^2),$$

and let $U = B_R(0)$ with $R < 1$. Then

$$u(x) = (x_2^2 - x_1^2)(-\log|x|)^{1/2}$$

belongs to $C(\bar{B}_R(0)) \cap C^\infty(\bar{B}_R(0)\backslash\{0\})$ and satisfies

$$\begin{cases} -\Delta u = f & \text{in } B_R(0)\backslash\{0\}, \\ u = g & \text{on } \partial B_R(0), \end{cases} \tag{1.44}$$

but u is not in $C^2(B_R(0))$ since we can check that $\lim_{|x|\to 0} D_{11}u = -\infty$. To see this, assume there exists such a classical solution v. Then $w = u - v$ is harmonic in $B_R(0)\backslash\{0\}$, but basic theory on removable singularities of harmonic functions, see Theorem 1.5, ensures that w can be redefined at the origin so that w is harmonic in $B_R(0)$. Thus, w is $C^2(B_R(0))$ and therefore u must also belong to $C^2(B_R(0))$. Hence, $\lim_{|x|\to 0} D_{11}u$ exists and we arrive at a contradiction.

In view of the above observations, we should assume the data f is Hölder continuous. We first introduce some definitions. Let x_0 be a point in $\mathbb{R}^n$ and f is a function defined on a bounded set U containing x_0.

Definition 1.8. *Let $\alpha \in (0,1)$. Then f is said to be* **Hölder continuous with exponent α at x_0** *if the quantity*

$$[f]_{\alpha;x_0} = \sup_U \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}$$

is finite. Here $[f]_{\alpha;x_0}$ **is called the α-Hölder coefficient of f at x_0 with respect to U.**

Moreover, f is said to be **uniformly Hölder continuous with exponent α in U** *if the quantity*

$$[f]_{\alpha;U} = \sup_{x,y\in U,\, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite.

Definition 1.9. *Likewise, f is said to be* **locally Hölder continuous with exponent α in U** *if f is uniformly Hölder continuous with exponent α on compact subsets of U. Obviously, the two notions of Hölder continuity coincide if U is a compact subset.*

Let $\alpha \in (0,1)$, $U \subset \mathbb{R}^n$ be an open set and k a non-negative integer.

Definition 1.10. *The* **Hölder spaces** *$C^{k,\alpha}(\bar{U})$ (respectively $C^{k,\alpha}(U)$) are defined as the subspaces of $C^k(\bar{U})$ (respectively $C^k(U)$) consisting of functions whose k^{th} order partial derivatives are uniformly Hölder continuous (respectively locally Hölder continuous) with exponent α in U. For short, we denote $C^{0,\alpha}(\bar{U})$ (respectively $C^{0,\alpha}(U)$) simply by $C^\alpha(\bar{U})$ (respectively $C^\alpha(U)$).*

Remark 1.13. *Let us discuss the endpoint cases for α. If $\alpha = 1$, $C^\alpha(\bar{U})$ (respectively $C^\alpha(U)$) is often called the space of uniformly Lipschitz continuous functions (respectively locally Lipschitz continuous functions). If $\alpha = 0$, $C^{k,0}(\bar{U})$ (respectively $C^{k,0}(U)$) are the usual C^k spaces. Moreover, for $\alpha \in [0,1]$, $C_0^{k,\alpha}(U)$ denotes the space of functions in $C^{k,\alpha}(U)$ having compact support in U.*

For $k = 0, 1, 2, \ldots$, consider the following seminorms

$$[u]_{k,0;U} = |D^k u|_{0;U} = \sup_{|\beta|=k} \sup_U |D^\beta u|,$$

$$[u]_{k,\alpha;U} = [D^k u]_{\alpha;U} = \sup_{|\beta|=k} [D^\beta u]_{\alpha,U}.$$

With these seminorms, we can define the norms

$$\|u\|_{C^k(\bar{U})} = |u|_{k;U} = |u|_{k,0;U} = \sum_{j=0}^{k} [u]_{j,0;U} = \sum_{j=0}^{k} |D^j u|_{0;U},$$
$$\|u\|_{C^{k,\alpha}(\bar{U})} = |u|_{k,\alpha;U} = |u|_{k;U} + [u]_{k,\alpha;U} = |u|_{k;U} + [D^k u]_{\alpha;U},$$

on the spaces $C^k(\bar{U})$, $C^{k,\alpha}(\bar{U})$. It is sometimes useful, especially in this section anyway, to consider non-dimensional norms on these spaces. In particular, if U is bounded with $d = diam(U)$, we set

$$\|u\|'_{C^k(\bar{U})} = |u|'_{k;U} = \sum_{j=0}^{k} d^j [u]_{j,0;U} = \sum_{j=0}^{k} d^j |D^j u|_{0;U},$$
$$\|u\|'_{C^{k,\alpha}(\bar{U})} = |u|'_{k,\alpha;U} = |u|'_{k;U} + d^{k+\alpha}[u]_{k,\alpha;U} = |u|'_{k;U} + d^{k+\alpha}[D^k u]_{\alpha;U}.$$

Not surprisingly, we have the following basic result, which we give without proof.

Theorem 1.22. *Let $\alpha \in [0,1]$ and $U \subset \mathbb{R}^n$ be an open domain. The spaces $C^k(\bar{U})$, $C^{k,\alpha}(\bar{U})$ equipped with the norms defined above are Banach spaces.*

The following algebra property holds: the product of Hölder continuous functions is again Hölder continuous. Namely, if $u \in C^\alpha(\bar{U})$, $v \in C^\beta(\bar{U})$, we have $uv \in C^\gamma(\bar{U})$ where $\gamma = \min\{\alpha, \beta\}$, and

$$\|uv\|_{C^\gamma(\bar{U})} \leq \max(1, d^{\alpha+\beta-2\gamma}) \|u\|_{C^\alpha(\bar{U})} \|v\|_{C^\beta(\bar{U})},$$
$$\|uv\|'_{C^\gamma(\bar{U})} \leq \|u\|'_{C^\alpha(\bar{U})} \|v\|'_{C^\beta(\bar{U})}.$$

1.4.1 The Dirichlet Problem for Poisson's Equation

We now develop the regularity properties of Newtonian potentials. We will use this to then show that Poisson's equation in a bounded domain U may be solved under the same boundary conditions for which Laplace's equation is solvable.

Lemma 1.5. *Let f be a bounded and integrable in U, and let w be the Newtonian potential of f. Then $w \in C^1(\mathbb{R}^n)$ and for any $x \in U$,*

$$D_i w(x) = \int_U D_i \Gamma(x-y) f(y)\, dy, \quad i = 1, 2, \dots, n.$$

Proof. It is easy to check the following derivative estimates for Γ:

$$\begin{cases} |D_i \Gamma(x-y)| \leq \dfrac{1}{n\omega_n} |x-y|^{1-n}, \\[2mm] |D_{ij} \Gamma(x-y)| \leq \dfrac{1}{\omega_n} |x-y|^{-n}, \\[2mm] |D^\beta \Gamma(x-y) \leq C(n, |\beta|) |x-y|^{2-n-|\beta|}. \end{cases} \tag{1.45}$$

From this, the function

$$v(x) = \int_U D_i\Gamma(x-y)f(y)\,dy$$

is well-defined. We now show that $v = D_i w$. To do so, for $\epsilon > 0$, let $\eta_\epsilon(x,y) = \eta(|x-y|/\epsilon)$ where $\eta = \eta(|x|)$ is some non-negative radial function in $C^1(\mathbb{R})$ with $supp(\eta) \subseteq [0,1]$, $supp(\eta') \subseteq [0,2]$, and

$$\eta(|x|) := \begin{cases} 0, & \text{if } |x| \le 1, \\ 1, & \text{if } |x| \ge 2. \end{cases}$$

Define

$$w_\epsilon(x) = \int_U \eta_\epsilon(x,y)\Gamma(x-y)f(y)\,dy,$$

which is obviously in $C^1(\mathbb{R}^n)$. Then, there holds,

$$v(x) - D_i w_\epsilon(x) = \int_{B_{2\epsilon}(x)} D_i\Big[(1-\eta_\epsilon(x,y))\Gamma(x-y)\Big]f(y)\,dy.$$

Hence, if $n \ge 3$,

$$|v(x) - D_i w_\epsilon(x)| \le \|f\|_\infty \int_{B_{2\epsilon}(x)} |D_i\Gamma(x-y)| + \frac{2}{\epsilon}|\Gamma(x-y)|\,dy \le \frac{2n\epsilon}{n-2}\|f\|_\infty.$$

Note that if $n = 2$, it follows that

$$|v(x) - D_i w_\epsilon(x)| \le 4\epsilon(1 + |\ln 2\epsilon|).$$

In either case, we conclude that as $\epsilon \longrightarrow 0$, w_ϵ and $D_i w_\epsilon$ converge uniformly on compact subsets of $\mathbb{R}^n$ to w and v, respectively. Therefore, $w \in C^1(\mathbb{R}^n)$ and $v = D_i w$. $\square$

Lemma 1.6. *Let f be bounded and locally Hölder continuous in U with exponent $\alpha \in (0,1]$, and let w be the Newtonian potential of f. Then*

(a) $w \in C^2(U)$;

(b) $-\Delta w = f$ in U;

(c) For any $x \in U$,

$$D_{ij}w(x) = \int_{U_0} D_{ij}\Gamma(x-y)(f(y)-f(x))\,dy - f(x)\int_{\partial U_0} D_i\Gamma(x-y)\nu_j(y)\,dS_y, \quad i,j = 1,2,\dots,n. \tag{1.46}$$

Here, U_0 is any domain containing U for which the divergence theorem holds and f is extended to vanish outside U.

Proof. Using the derivative estimates of (1.45) for $D^2\Gamma$ and since f is pointwise Hölder cotinuous in U, the function

$$u(x) = \int_{U_0} D_{ij}\Gamma(x-y)(f(y)-f(x))\,dy - f(x)\int_{\partial U_0} D_i\Gamma(x-y)\nu_j(y)\,dS_y,$$

is well-defined. Let $v = D_i w$ and define for $\epsilon > 0$,

$$v_\epsilon(x) = \int_U D_i\Gamma(x-y)\eta_\epsilon(x,y)f(y)\,dy,$$

where η_ϵ is the same test function as in the previous lemma. Obviously, $v_\epsilon \in C^1(U)$ and for $\epsilon > 0$ sufficiently small, differentiating leads to

$$\begin{aligned}
D_j v_\epsilon(x) &= \int_U D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))f(y)\,dy \\
&= \int_U D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))(f(y)-f(x))\,dy + f(x)\int_{U_0} D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))\,dy \\
&= \int_U D_j(D_i\Gamma(x-y)\eta_\epsilon(x,y))(f(y)-f(x))\,dy + f(x)\int_{\partial U_0} D_i\Gamma(x-y)\nu_j(y)\,dS_y.
\end{aligned}$$

Hence, by subtracting this from $u(x)$, we estimate that

$$\begin{aligned}
|u(x) - D_j v_\epsilon(x)| &= \Big|\int_{B_{2\epsilon}(x)} D_j[(1-\eta_\epsilon)D_i\Gamma(x-y)](f(y)-f(x))\,dy\Big| \\
&\le [f]_{\alpha;x}\int_{B_{2\epsilon}(x)} \Big(|D_{ij}\Gamma| + \frac{2}{\epsilon}|D_i\Gamma|\Big)|x-y|^\alpha\,dy \\
&\le \Big(\frac{n}{\alpha} + 4\Big)(2\epsilon)^\alpha[f]_{\alpha;x},
\end{aligned}$$

provided that $2\epsilon < dist(x,\partial U)$. Therefore, $D_j v_\epsilon$ converges to u uniformly on compact subsets of U as $\epsilon \longrightarrow 0$. Of course, v_ϵ converges to $v = D_i w$ as $\epsilon \longrightarrow 0$. Hence, $w \in C^2(U)$ and $u = D_{ij}w$. Then, if we set $U_0 = B_r(x)$ for r suitably large,

$$-\Delta w(x) = \frac{1}{\omega_n r^{n-1}} f(x)\int_{\partial B_r(x)} \nu_i(y)\nu_i(y)\,dS_y = f(x).$$

This completes the proof of the lemma. $\square$

A consequence of Lemmas 1.5 and 1.6 is the following theorem. This result should be compared with Theorem 1.15 as it generalizes that result in that f is assumed to be bounded and locally Hölder continuous in U rather than the stronger condition that $f \in C_c^2(U)$.

Theorem 1.23. *Let U be a bounded domain and suppose that each point of ∂U is regular (with respect to the Laplacian). Then, if f is a bounded, locally Hölder continuous function in U, the classical Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ \quad\; u = g & \text{on } \partial U, \end{cases} \tag{1.47}$$

is uniquely solvable for any continuous boundary values g in the class of classical solutions, i.e., $u \in C^2(U) \cap C(\bar{U})$.

Proof. Let w be the Newtonian potential of f and consider the function $v = u - w$. It is clear that $-\Delta v = 0$ in U and $v = g - w$ on ∂U, but it is obvious that the unique solvability of this boundary-value problem for Laplace's equation will imply the desired result. Now, the existence of classical solutions of Laplace's equation follows from several methods, e.g. the Perron method, which are provided in the next chapter, and the uniqueness of the solution is a consequence of the maximum principles. □

Remark 1.14. *Here, a boundary point will be called regular (with respect to the Laplacian) if there exists a barrier function at that point. For the definition of a barrier function, see* (2.22) *in the next chapter discussing Perron's method. There we shall see that if ∂U is C^2 then each point on the boundary is indeed regular. Furthermore, the regularity theory below indicates that the unique solution of the above Dirichlet problem on a Euclidean ball domain belongs to $C^{2,\alpha}(U) \cap C(\bar{U})$*

Remark 1.15. *If $U = B_R(0)$, the last theorem follows from the two preceding lemmas and Poisson's formula* (1.43) *for the ball. In fact, we even have an explicit representation of the unique solution, which is given by*

$$u(x) = \int_{\partial B_R(0)} K(x,y) g(y)\, dS_y + \int_{B_R(0)} G(x,y) f(y)\, dy,$$

where $K(x,y)$ and $G(x,y) = \Gamma(y-x) - \phi^x(y)$ are Poisson's kernel and the Green's function on the ball, respectively. In particular, for all $x, y \in B_R(0)$, $x \neq y$,

$$G(x,y) = \Gamma(\sqrt{(|x||y|/2)^2 + R^2 - 2x \cdot y}) - \Gamma(\sqrt{|x|^2 + |y|^2 - 2x \cdot y}). \tag{1.48}$$

1.4.2 Interior Hölder Estimates for Second Derivatives

For concentric balls of radius $R > 0$ centered at x_0 in $\mathbb{R}^n$, we set $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0)$.

Lemma 1.7. *Suppose that $f \in C^\alpha(\bar{B}_2)$, $\alpha \in (0,1)$, and let w be the Newtonian potential of f in B_2. Then $w \in C^{2,\alpha}(\bar{B}_1)$ and*

$$\begin{aligned}
&|D^2 w|'_{0,\alpha;B_1} \leq C(n,\alpha) |f|'_{0,\alpha;B_2}, \\
&|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \leq C(n,\alpha)(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).
\end{aligned}$$

Remark 1.16. *For general domains $U_1 \subset B_1(x_0)$ and $B_2(x_0) \subset U_2$, and $f \in C^\alpha(\bar{U}_2)$ and w is the Newtonian potential of f over U_2. Then the statement of Lemma 1.7 with U_i replacing $B_i(x_0)$, $i = 1, 2$, respectively, still remains true.*

Proof of Lemma 1.7. For any $x \in B_1$, identity (1.46) yields

$$D_{ij}w(x) = \int_{B_2} D_{ij}\Gamma(x-y)[f(y)-f(x)]\,dy - f(x)\int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)\,dS_y \tag{1.49}$$

and thus, by the derivative estimates in (1.45),

$$\begin{aligned}|D_{ij}w(x)| &\le \frac{|f(x)|}{n\omega_n}R^{1-n}\int_{\partial B_2} dS_y + \frac{[f]_{\alpha;x}}{\omega_n}\int_{B_2}|x-y|^{\alpha-n}\,dy \\ &= 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha[f]_{\alpha;x} \le C(n,\alpha)(|f(x)| + R^\alpha[f]_{\alpha;x}).\end{aligned} \tag{1.50}$$

Then, again (1.46) implies that for any other point $\bar{x} \in B_1$ we have

$$D_{ij}w(\bar{x}) = \int_{B_2} D_{ij}\Gamma(\bar{x}-y)[f(y)-f(\bar{x})]\,dy - f(\bar{x})\int_{\partial B_2} D_i\Gamma(\bar{x}-y)\nu_j(y)\,dS_y. \tag{1.51}$$

Set $\delta = |x-\bar{x}|$ and $\xi = (x+\bar{x})/2$. Subtracting (1.51) from (1.49) yields

$$D_{ij}w(x) - D_{ij}w(\bar{x}) = f(x)I_1 + [f(x)-f(\bar{x})]I_2 + I_3 + I_4 + [f(x)-f(\bar{x})]I_5 + I_6,$$

where

$$\begin{aligned}
I_1 &= \int_{\partial B_2} [D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)]\nu_j(y)\,dS_y,\\
I_2 &= \int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)\,dS_y,\\
I_3 &= \int_{B_\delta(\xi)} D_{ij}\Gamma(x-y)[f(x)-f(y)]\,dy,\\
I_4 &= \int_{B_\delta(\xi)} D_{ij}\Gamma(\bar{x}-y)[f(y)-f(\bar{x})]\,dy,\\
I_5 &= \int_{B_2\setminus B_\delta(\xi)} D_{ij}\Gamma(x-y)\,dy,\\
I_6 &= \int_{B_2\setminus B_{\delta(\xi)}} [D_{ij}\Gamma(x-y) - D_{ij}\Gamma(\bar{x}-y)][f(\bar{x})-f(y)]\,dy.
\end{aligned}$$

We estimate each term I_i: For some $\tilde{x}$ between x and $\bar{x}$,

$$\begin{aligned}
|I_1| &\le |x-\bar{x}|\int_{\partial B_2}|DD_i\Gamma(\tilde{x}-y)|\,dS_y\\
&\le \frac{n^2 2^{n-1}|x-\bar{x}|}{R} \text{ (since } |\tilde{x}-y| \ge R \text{ for } y \in \partial B_2)\\
&\le n^2 2^{n-\alpha}\Big(\frac{\delta}{R}\Big)^\alpha \text{ (since } \delta = |x-\bar{x}| < 2R).
\end{aligned}$$

$$|I_2| \le \frac{1}{n\omega_n} R^{1-n} \int_{\partial B_2} dS_y = 2^{n-1}.$$

$$|I_3| \le \int_{B_\delta(\xi)} |D_{ij}\Gamma(x-y)||f(x)-f(y)|\,dy \le \frac{1}{\omega_n}[f]_{\alpha;x} \int_{B_{(3/2)\delta}(x)} |x-y|^{\alpha-n}\,dy \le \frac{n}{\alpha}\Big(\frac{3\delta}{2}\Big)^{\alpha}[f]_{\alpha;x}.$$

Similarly,

$$|I_4| \le \frac{n}{\alpha}\Big(\frac{3\delta}{2}\Big)^{\alpha}[f]_{\alpha;\bar{x}}.$$

$$\begin{aligned}
|I_5| &= \Big|\int_{\partial(B_2\setminus B_\delta(\xi))} D_i\Gamma(x-y)\nu_j(y)\,dS_y\Big| \\
&\le \Big|\int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)\,dS_y\Big| + \Big|\int_{\partial B_\delta(\xi)} D_i\Gamma(x-y)\nu_j(y)\,dS_y\Big| \\
&\le 2^{n-1} + \frac{1}{n\omega_n}\Big(\frac{\delta}{2}\Big)^{1-n}\int_{\partial B_\delta(\xi)} dS_y = 2^{n-1}+2^{n-1} = 2^n.
\end{aligned}$$

$$\begin{aligned}
|I_6| &\le |x-\bar{x}| \int_{B_2\setminus B_\delta(\xi)} |DD_{ij}\Gamma(\tilde{x}-y)||f(\bar{x})-f(y)|\,dy \text{ (for some } \tilde{x} \text{ between } x \text{ and } \bar{x}) \\
&\le c(n)\delta \int_{|y-\xi|\ge\delta} \frac{|f(\bar{x})-f(y)|}{|\tilde{x}-y|^{n+1}}\,dy \\
&\le c\delta[f]_{\alpha;\bar{x}} \int_{|y-\xi|\ge\delta} |\xi-y|^{\alpha-n-1}\,dy \text{ (since } |\bar{x}-y| \le (3/2)|\xi-y| \le 3|\tilde{x}-y|) \\
&\le c(n)(1-\alpha)^{-1}2^{n+1}\Big(\frac{3}{2}\Big)^{\alpha}\delta^{\alpha}[f]_{\alpha;\bar{x}}.
\end{aligned}$$

Combining these estimates gives us

$$|D_{ij}w(\bar{x}) - D_{ij}w(x)| \le C(n,\alpha)\Big(R^{-\alpha}|f(x)| + [f]_{\alpha;x} + [f]_{\alpha;\bar{x}}\Big)|x-\bar{x}|^{\alpha}.$$

Hence, this along with (1.50) completes the proof of the lemma. □

Theorem 1.24. *Let $f \in C_0^{\alpha}(\mathbb{R}^n)$ and suppose $u \in C_0^2(\mathbb{R}^n)$ satisfy Poisson's equation,*

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

Then $u \in C_0^{2,\alpha}(\mathbb{R}^n)$, and if $B = B_R(x_0)$ is any ball containing the support of u, then

$$\begin{aligned}
|D^2u|'_{0,\alpha;B} &\le C(n,\alpha)|f|'_{0,\alpha;B}, \\
|u|'_{1,B} &\le C(n)R^2|f|_{0;B}.
\end{aligned}$$

Proof. As indicated in Theorem 1.15 or Lemma 1.6, we can conclude that $u = \Gamma * f$, even if it was assumed there that $f \in C_c^2(\mathbb{R}^n)$ as it still holds true even when $f \in C_0^\alpha(\mathbb{R}^n)$. The estimates for Du and D^2u follow, respectively, from Lemma 1.5 and Lemma 1.7 and the fact that f has compact support. The estimate for $|u|_{0;B}$ follows at once from that for Du. □

The restriction that u and f have compact support in the last theorem can be removed.

Theorem 1.25. *Let U be a domain in $\mathbb{R}^n$ and let $f \in C^\alpha(U)$, $\alpha \in (0,1)$, and let $u \in C^2(U)$ satisfy Poisson's equation, $-\Delta u = f$ in U. Then $u \in C^{2,\alpha}(U)$ and for any two concentric balls $B_R(x_0)$, $B_{2R}(x_0) \subset\subset U$, we have*

$$|u|'_{2,\alpha;B_R(x_0)} \leq C(n,\alpha)(|u|_{0;B_{2R}(x_0)} + R^2|f|'_{0,\alpha;B_{2R}(x_0)}). \tag{1.52}$$

A consequence of the interior estimate (1.52) is the equicontinuity on compact subsets of the second derivatives of any bounded set of solutions of Poisson's equation. Therefore, the Arzelà–Ascoli theorem implies the following result on the compactness of solutions to Poisson's equation.

Corollary 1.5. *Any bounded sequence of solutions of Poisson's equation, $-\Delta u = f$ in U, where $f \in C^\alpha(U)$, contains a subsequence converging uniformly on compact subsets of U to another solution.*

As a consequence of this compactness result, we establish an existence result for the Dirichlet problem. Here, we denote $d_x = d_x(U) = dist(x, \partial U)$.

Theorem 1.26. *Let B be a ball in $\mathbb{R}^n$ and f be a function in $C^\alpha(B)$ for which*

$$\sup_{x\in B} d_x^{2-\beta}|f(x)| \leq N < \infty$$

for some $\beta \in (0,1)$. Then there exists a unique function $u \in C^2(B) \cap C(\bar{B})$ satisfying

$$\begin{cases} -\Delta u = f & \text{in } B, \\ \quad\;\; u = 0 & \text{on } \partial B. \end{cases}$$

Furthermore, the solution u satisfies the estimate

$$\sup_{x\in B} d_x^{-\beta}|u(x)| \leq CN, \tag{1.53}$$

where $C = C(\beta)$.

Proof. Step 1: Estimate (1.53) follows from a simple barrier argument, i.e., let $B = B_R(x_0)$, $r = |x - x_0|$ and set $w(x) = (R^2 - r^2)^\beta$. A direct calculation will show that

$$\begin{aligned} \Delta w(x) = &-2\beta(R^2 - r^2)^{\beta-2}[n(R^2 - r^2) + 2(1-\beta)r^2] \\ \leq &-4\beta(1-\beta)R^2(R^2 - r^2)^{\beta-2} \leq -\beta(1-\beta)R^\beta(R-r)^{\beta-2}. \end{aligned}$$

Now suppose that $-\Delta u = f$ in B and $u = 0$ on ∂B. Since $d_x = R - r$, the hypothesis yields

$$|f(x)| \leq N d_x^{\beta-2} = N(R-r)^{\beta-2} \leq -C_0 N \Delta w,$$

where $C_0 = [\beta(1-\beta)R^\beta]^{-1}$. Hence,

$$-\Delta(C_0 N w \pm u) \geq 0 \text{ in } B, \text{ and } C_0 N w \pm u = 0 \text{ on } \partial B.$$

Therefore, the maximum principle implies

$$|u(x)| \leq C_0 N w(x) \leq C N d_x^\beta \quad \text{for } x \in B, \tag{1.54}$$

which implies (1.53) with constant $C = 2/\beta(1-\beta)$.

Step 2: We now prove the existence of u. Define

$$f_m = \begin{cases} m, & \text{if } f \geq m, \\ f, & \text{if } |f| \leq m, \\ -m, & \text{if } f \leq -m, \end{cases}$$

and let $\{B_k\}$ be a sequence of concentric balls exhausting B such that $|f| \leq k$ in B_k. We define u_m to be the solution of $-\Delta u_m = f_m$ in B and $u_m = 0$ on ∂B. By (1.53),

$$\sup_{x \in B} d_x^{-\beta} |u_m(x)| \leq C \sup_{x \in B} d_x^{2-\beta} |f_m(x)| \leq CN,$$

so that the sequence $\{u_m\}$ is uniformly bounded and $-\Delta u_m = f$ in B_k for $m \geq k$. Hence, by Corollary 1.5 applied successively to the sequence of balls B_k, we can extract a convergent subsequence of $\{u_m\}$ with limit point u in $C^2(B)$ satisfying $-\Delta u = f$ in B. Moreover, there holds $|u(x)| \leq C N d_x^\beta$ and so $u = 0$ on ∂B. This completes the proof of the theorem. □

1.4.3 Boundary Hölder Estimates for Second Derivatives

We may refine the interior Hölder regularity estimates by extending them up to the boundary. We focus only on ball domains but the results certainly apply to bounded and open domains with smooth boundary. We refer the reader to Chapter 3 for more details on obtaining regularity estimates up to the boundary for general smooth domains.

We start with some notation. Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$ be the usual upper half-space with boundary $T = \partial \mathbb{R}^n_+$, $B_2 := B_{2R}(x_0)$, $B_1 = B_R(x_0)$ where $R > 0$ and $x_0 \in \mathbb{R}^n_+$. Moreover, set $B_2^+ := B_2 \cap \mathbb{R}^n_+$ and $B_1^+ = B_1 \cap \mathbb{R}^n_+$.

Lemma 1.8. *Let $f \in C^\alpha(\bar{B}_2^+)$ and let w be the Newtonian potential of f in B_2^+. Then $w \in C^{2,\alpha}(\bar{B}_1^+)$ and*

$$|D^2 w|'_{0,\alpha;B_1^+} \leq C |f|'_{0,\alpha;B_2^+} \tag{1.55}$$

where $C = C(n, \alpha)$.

Proof. We may assume B_2 intersects T, otherwise the result is already contained in Lemma 1.7. The representation (1.46) holds for $D_{ij}w$ within $U_0 = B_2^+$. If either i or $j \neq n$, then the portion of the boundary integral

$$\int_{\partial B_2^+} D_i\Gamma(x-y)\nu_j(y)\,dS_y = \int_{\partial B_2^+} D_j\Gamma(x-y)\nu_i(y)\,dS_y$$

on T vanishes since ν_i or ν_j equals to 0 there. The estimates in Lemma 1.7 for $D_{ij}w$ (i or $j \neq 0$) then proceed exactly as before with B_2 replaced with B_2^+, $B_\delta(\xi)$ replaced by $B_\delta(\xi)\cap B_2^+$ and ∂B_2 replaced by $\partial B_2^+\backslash T$. Finally, $D_{nn}w$ can be estimated from the equation $-\Delta w = f$ and the estimates $D_{kk}w$ for $k = 1, 2, \ldots, n-1$. $\square$

Theorem 1.27. *Let $u \in C^2(B_2^+)\cap C(\bar{B}_2^+)$, $f \in C^\alpha(\bar{B}_2^+)$, satisfy $-\Delta u = f$ in B_2^+, $u = 0$ on T. Then $u \in C^{2,\alpha}(\bar{B}_1^+)$ and we have*

$$|u|'_{2,\alpha;B_1^+} \leq C(|u|_{0;B_2^+} + R^2|f|'_{0,\alpha;B_2^+}) \tag{1.56}$$

where $C = C(n,\alpha)$.

Proof. Let $x' = (x_1, x_2, \ldots, x_{n-1})$, $x^* = (x', -x_n)$ and define

$$f^*(x) = f^*(x', x_n) := \begin{cases} f(x', x_n), & \text{if } x_n \geq 0, \\ f(x', -x_n), & \text{if } x_n \leq 0. \end{cases}$$

We assume that B_2 intersects T; otherwise Theorem 1.7 already implies estimate (1.56). Now set $B_2^- := \{x \in \mathbb{R}^n \,|\, x^* \in B_2^+\}$ and $D = B_2^+ \cup B_2^- \cup (B_2 \cap T)$. Then $f^* \in C^\alpha(\bar{D})$ and $|f^*|'_{0,\alpha;B_2^+}$.

Define

$$\begin{aligned} w(x) &= \int_{B_2^+} [\Gamma(x-y) - \Gamma(x^*-y)]f(y)\,dy \\ &= \int_{B_2^+} [\Gamma(x-y) - \Gamma(x-y^*)]f(y)\,dy, \end{aligned} \tag{1.57}$$

so that $w(x', 0) = 0$ and $-\Delta w = f$ in B_2^+. Observe that

$$\int_{B_2^+} \Gamma(x-y^*)f(y)\,dy = \int_{B_2^-} \Gamma(x-y)f^*(y)\,dy,$$

so then we get

$$w(x) = 2\int_{B_2^+} \Gamma(x-y)f(y)\,dy - \int_D \Gamma(x-y)f^*(y)\,dy.$$

Letting

$$w^*(x) = \int_D \Gamma(x-y)f^*(y)\,dy,$$

the remark following Lemma 1.7 with $U_1 = B_1^+$ and $U_2 = D$ implies that

$$|D^2 w^*|'_{0,\alpha;B_1^+} \le C|f^*|'_{0,\alpha;D} \le 2C|f|'_{0,\alpha;B_2^+}.$$

Combining this with Lemma 1.8 yields

$$|D^2 w|'_{0,\alpha;B_1^+} \le C|f|'_{0,\alpha;B_2^+}.$$

Now let $v = u - w$, then $\Delta v = 0$ in B_2^+ and $v = 0$ on T. By reflection, we may extend v to a harmonic function in B_2 and thus estimate (1.56) follows from the interior derivative estimate for harmonic functions (cf. Theorem 2.10 in [11]). □

Theorem 1.28. *Let B be a ball in $\mathbb{R}^n$ and u and f functions on $\bar{B}$ satisfying $u \in C^2(B) \cap C(\bar{B})$, $f \in C^\alpha(\bar{B})$ and*

$$\begin{cases} -\Delta u = f & \text{in } B, \\ \quad u = 0 & \text{on } \partial B, \end{cases}$$

then $u \in C^{2,\alpha}(\bar{B})$.

Proof. By translation invariance, we may assume ∂B passes through the origin. The inversion mapping $x \mapsto x^* := x/|x|^2$ is a bicontinuous and smooth mapping of the punctured space $\mathbb{R}^n\backslash\{0\}$ onto itself which maps B onto a half-space, B^*. Moreover, since $u \in C^2(B) \cap C(\bar{B})$, the Kelvin transform of u, i.e.,

$$v(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big),$$

belongs to $C^2(B^*) \cap C(\bar{B}^*)$ and satisfies

$$-\Delta v(x^*) = |x^*|^{-(n+2)} f\Big(\frac{x^*}{|x^*|^2}\Big), \; x \in B^*.$$

Hence, we can apply Theorem 1.27 to the Kelvin transform v and since by translation invariance any point of ∂B may be re-centered to be the origin, we conclude that $u \in C^{2,\alpha}(\bar{B})$. □

We conclude now with an application of the boundary estimates to obtain an existence result for the Dirichlet problem.

Corollary 1.6. *Let $\varphi \in C^{2,\alpha}(\bar{B})$, $f \in C^\alpha(\bar{B})$. Then the Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } B, \\ \quad u = \varphi & \text{on } \partial B, \end{cases}$$

is uniquely solvable for a function $u \in C^{2,\alpha}(\bar{B})$.

Proof. Writing $v = u - \varphi$, the problem is reduced to solving the problem

$$\begin{cases} -\Delta v = f - \Delta\varphi & \text{in } B, \\ \quad v = 0 & \text{on } \partial B, \end{cases}$$

which is solvable for $v \in C^2(B) \cap C(\bar{B})$ by the usual representation formula via Green's functions and consequently for $v \in C^{2,\alpha}(\bar{B})$ by Theorem 1.28. □

CHAPTER 2

Existence Theory

2.1 The Lax-Milgram Theorem

Theorem 2.1 (Lax–Milgram). *Let H be a Hilbert space with norm $\|\cdot\|$ and $B : H \times H \longrightarrow \mathbb{R}$ is a bilinear form. Suppose that there exist numbers α, $\beta > 0$ such that for any $u, v \in H$*

(i) Boundedness: $|B[u, v]| \leq \alpha \|u\| \cdot \|v\|$,

(ii) Coercivity: $\beta \|u\|^2 \leq B[u, u]$,

then for each $f \in L^2(U)$ there exists a unique $u \in H$ such that

$$B[u, v] = (f, v) \text{ for all } v \in H.$$

To prove the theorem, we first recall the Riesz representation theorem for Hilbert spaces.

Theorem 2.2 (Riesz representation). *If f is a bounded linear functional on a Hilbert Space H with inner product $(\cdot, \cdot)$, then there exists an element $v \in H$ such that $< f, u >= (v, u)$ for all $u \in H$.*

It is clear that the inner product is a bilinear form which satisfies both the requirements of the Lax–Milgram theorem. However, the Lax–Milgram theorem is a stronger result than the Riesz representation theorem in that it does not require the bilinear form to be symmetric.

Proof. Existence: For each fixed $w \in H$, $v \longrightarrow B[w, v]$ is a bounded linear functional on H. By the Riesz representation theorem, there exists a $u \in H$ such that $(u, v) = B[w, v]$ for all $v \in H$. We define the operator $A : H \longrightarrow H$ by $u = A[w]$.

Step 1: Claim that $A : H \longrightarrow H$ is a bounded linear operator: To prove A is linear, observe that

$$\begin{aligned}(A[\lambda_1 u_1 + \lambda_2 u_2],\, v) &= B[\lambda_1 u_1 + \lambda_2 u_2,\, v] = \lambda_1 B[u_1,\, v] + \lambda_2 B[u_2,\, v] \\ &= (\lambda_1 A[u_1] + \lambda_2 A[u_2],\, v) \text{ for all } v \in H.\end{aligned}$$

Thus, $A[\lambda_1 u_1 + \lambda_2 u_2] = \lambda_1 A[u_1] + \lambda_2 A[u_2]$.

Moreover, A is bounded since

$$\|Au\|^2 = (Au,\, Au) = B[Au, u] \leq \alpha \|u\| \cdot \|Au\|$$

Hence, $\|Au\| \leq \alpha \|u\|$.

Step 2: Claim $Ran(A)$ is closed in H.
Let $\{y_k\}$ be a convergent sequence in $ran(A)$ so that there is a sequence $\{u_k\} \subset H$ for which $y_k = A[u_k] \longrightarrow y \in H$. By coercivity, $\|u_k - u_j\| \leq \beta \|A[u_k] - A[u_j]\|$, which implies $\{u_k\}$ is a Cauchy sequence in H. Hence, u_k converges to some element $u \in H$ and $y = A[u]$; that is, $y \in Ran(A)$, thereby proving $Ran(A)$ is closed in H.

Step 3: Claim $Ran(A) = H$.
On the contrary, assume that $Ran(A) \neq H$. Thus, we have that $H = ran(A) \oplus ran(A)^\perp$ since $Ran(A)$ is closed, and we choose a non-zero element $z \in ran(A)^\perp$. By the coercivity condition, $\beta \|z\|^2 \leq B[z, z] = (Az, z) = 0$ and we arrive at a contradiction.

Step 4: For each $f \in L^2$, the Riesz representation theorem once again implies there exists an element $z \in H$ for which $(z, v) = < f, v >$ for all $v \in H$. In turn, we can find a u such that $z = A[u]$, i.e., $(z, v) = (Au, v) = B[u,\, v]$ for all $v \in H$. Hence, we have found an element $u \in H$ for which $B[u,\, v] = (f, v)$ for all $v \in H$.

Uniqueness: Suppose that u_1 and u_2 are two such elements satisfying $B[u_1,\, v] = (f, v)$ and $B[u_2,\, v] = (f, v)$ for all $v \in H$, respectively. This implies that $B[u_1 - u_2, v] = 0$ for all $v \in H$. Now, if $v = u_1 - u_2$, the coercivity condition implies $\beta \|u_1 - u_2\|^2 \leq B[u_1 - u_2,\, u_1 - u_2] = 0$. Hence, $u_1 = u_2$. $\square$

2.1.1 Existence of Weak Solutions

Our goal here is to prove existence and uniqueness of weak solutions to the Dirichlet boundary value problem of the following form:

$$\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.1}$$

where μ is a non-negative constant to be determined later. Developing this result relies mainly on certain energy estimates and the Lax-Milgram theorem. In addition, we will now focus strictly on the second order differential operator in divergence form with its associated bilinear form

$$B[u,\, v] := \int_U \sum_{i,j=1}^n a^{ij}(x) D_i u D_j v + \sum_{i=1}^n b^i(x) D_i u v + c(x) uv \, dx,$$

and assume that $a^{ij}, b^i, c \in L^\infty(U)$ for $i, j = 1, ..., n$. Furthermore, assume U is an open and bounded subset of $\mathbb{R}^n$ and denote $H := H_0^1(U)$.

Energy Estimates

Theorem 2.3. *There exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

(i) $|B[u,\, v]| \leq \alpha \|u\|_H \|v\|_H$

(ii) $\beta \|u\|_H^2 \leq B[u, u] + \gamma \|u\|_{L^2(U)}^2$ *for all* $u, v \in H$.

Proof. We prove the first estimate of the theorem.

$$\begin{aligned} |B[u,\, v]| &= \Big| \int_U \sum_{i,j=1}^n a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b^i(x) u_{x_i} v + cuv \, dx \Big| \\ &\leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty} \int_U |Du \cdot Dv| \, dx + \sum_{i=1}^n \|b^i\|_{L^\infty} \int_U |Du||v| \, dx + \|c\|_{L^\infty} \int_U |u||v| \, dx, \end{aligned}$$

since it was assumed that $a^{ij}, b^i, c \in L^\infty(U)$. Now apply Hölder's inequality sufficiently many times and use the definition of the H-norm to get

$$|B[u,\, v]| \leq C \|u\|_H \|v\|_H$$

for some constant C.

To prove the second part, the definition of (uniform) ellipticity will be used. By uniform ellipticity, there is some $\lambda > 0$ such that

$$\begin{aligned} \lambda \int_U |Du|^2 \, dx \leq \int_U \sum_{i,j=1}^n a^{ij}(x) u_{x_i} u_{x_j} \, dx &= B[u, u] - \int_U \sum_{i=1}^n b^i(x) u_{x_i} u + cu^2 \, dx \\ &\leq B[u, u] + \sum_{i=1}^n \|b^i\|_{L^\infty(U)} \int_U |Du||u| \, dx + \|c\|_{L^\infty(U)} \int_U u^2 \, dx \qquad (2.2) \end{aligned}$$

Using the Cauchy's inequality with ϵ i.e $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, a, b > 0, \epsilon > 0$, we have

$$|Du||u| \leq \epsilon |Du|^2 + \frac{u^2}{4\epsilon} \Longrightarrow \int_U |Du||u| \, dx \leq \epsilon \int_U |Du|^2 \, dx + \frac{1}{4\epsilon} \int_U u^2 \, dx.$$

We may choose $\epsilon > 0$ such that $\epsilon \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} < \frac{\lambda}{2}$, then plugging this back into (2.2) yields

$$\begin{aligned}\lambda \int_U |Du|^2\,dx &\leq B[u,u] + (\sum_{i=1}^{n} \|b^i\|_{L^\infty(U)})(\epsilon \int_U |Du|^2\,dx + \frac{1}{4\epsilon}\int_U u^2\,dx) + \|c\|_{L^\infty(U)} \int_U u^2\,dx \\ &\leq B[u,u] + \frac{\lambda}{2}\int_U |Du|^2\,dx + (\sum_{i=1}^{n} \|b^i\|_{L^\infty(U)})\frac{1}{4\epsilon} + \|c\|_{L^\infty(U)} \int_U u^2\,dx.\end{aligned}$$

Now some rearrangement of terms yields

$$\frac{\lambda}{2}\int_U |Du|^2\,dx \leq B[u,u] + C\int_U u^2\,dx.$$

Adding $\frac{\lambda}{2}\int_U |u|^2\,dx$ on both sides of this inequality gives us our desired result,

$$\frac{\lambda}{2}\|u\|_H^2 \leq B[u,u] + \left(C + \frac{\lambda}{2}\right)\|u\|_{L^2(U)}^2.$$

□

Remark 2.1. *From our estimate (ii), we see that $B[\cdot,\cdot]$ does not directly satisfy the hypotheses of the Lax-Milgram theorem whenever $\gamma > 0$. Our next theorem will take this into consideration as it provides our existence and uniqueness result for the Dirichlet boundary value problem.*

Theorem 2.4 (First Existence Theorem for weak solutions)**.** *There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H = H_0^1(U)$ of the Dirichlet boundary value problem*

$$\begin{cases} Lu + \mu u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.3}$$

Proof. Let γ be the same from the previous theorem, let $\mu \geq \gamma$ and define the bilinear form $B_\mu[u,v] = B[u,\,v] + \mu(u,v)_{L^2}$ with $u, v \in H$.
Claim: The bilinear form $B_\mu[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram theorem. More precisely, we have the bilinear estimate,

$$\begin{aligned}|B_\mu[u,v]| = |B[u,\,v] + \mu(u,v)_{L^2}| &\leq |B[u,\,v]| + \mu|(u,v)_{L^2}| \\ &\leq C\|u\|_H\|v\|_H + \mu\|u\|_{L^2}\|v\|_{L^2} \\ &\leq C\|u\|_H\|v\|_H,\end{aligned}$$

where in the second line we used the previous theorem and the Cauchy-Swharz inequality. Moreover, we have the coercivity estimate,

$$\begin{aligned}B_\mu[u,u] &= B[u,u] + \mu(u,u)_{L^2} \\ &\geq B[u,u] + \gamma(u,u)_{L^2} \\ &\geq C\|u\|_H,\end{aligned}$$

where we used the second bound from the energy estimates.

Now fix $f \in L^2(U)$ and set $\varphi_f(v) = (f, v)_{L^2}$. This is a bounded linear functional since, by the Cauchy-Schwarz inequality,

$$|\varphi_f(v)| = |(f, v)_{L^2}| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_H.$$

Thus, by the Lax-Milgram theorem, we can find a unique $u \in H$ satisfying $B_\mu[u, v] = \varphi_f(v)$ for all $v \in H$. That is, $u \in H$ is a unique weak solution to the Dirichlet boundary value problem. □

2.2 The Fredholm Alternative

First, we recall the Fredholm theory for compact operators then apply it to further develop our existence theory for second-order elliptic equations. Let X and Y be Banach spaces, H denotes a real Hilbert space with inner product $(\cdot, \cdot)$, and the operator L is the usual second order elliptic operator in divergence form.

Definition 2.1. *A bounded linear operator $K : X \longrightarrow Y$ is called* **compact** *provided each bounded sequence $\{u_k\}_{k=1}^\infty \subset X$, the sequence $\{Ku_k\}_{k=1}^\infty$ is precompact in Y, i.e., there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ such that $\{Ku_{k_j}\}_{j=1}^\infty$ converges in Y.*

Theorem 2.5 (Fredholm Alternative)**.** *Let $K : H \longrightarrow H$ be a compact linear operator. Then*

(a) The kernel $N(I - K)$ is finite dimensional,

(b) The range $R(I - K)$ is closed,

(c) $R(I - K) = N(I - K^*)^\perp$,

(d) $N(I - K) = \{0\}$ if and only if $R(I - K) = H$.

Remark 2.2. *This theorem basically asserts the following dichotomy, i.e., either*

(α) For each $f \in H$, the equation $u - Ku = f$ has a unique solution; or else

(β) the homogeneous equation $u - Ku = 0$ has non-trivial solutions.

Further, should (β) hold, the space of solutions of this homogeneous equation is finite dimensional, and the non-homogeneous equation

(γ) $u - Ku = f$ has a solution if and only if $f \in N(I - K^*)^\perp$.

We shall also require the following basic result on the spectrum of compact linear operators.

Theorem 2.6 (Spectrum of a compact operator)**.** *Assume* $dim(H) = \infty$ *and* $K : H \longrightarrow H$ *is a compact linear operator. Then*

(i) $0 \in \sigma(K)$,

(ii) $\sigma(K)\backslash\{0\} = \sigma_p(K)\backslash\{0\}$,

(iii) $\sigma(K)\backslash\{0\}$ *is finite, or else is a sequence tending to* 0.

2.2.1 Existence of Weak Solutions

Definition 2.2. *We define the following.*

(a) The operator L^**, the formal adjoint of* L*, is*

$$L^*v := -\sum_{i,j=1}^{n}(a^{ij}(x)v_{x_j})_{x_i} - \sum_{i=1}^{n} b^i(x)v_{x_i} + \Big(c(x) - \sum_{i=1}^{n} b^i_{x_i}(x)\Big)v,$$

provided $b^i \in C^1(\bar{U})$, $i = 1, 2, \ldots, n$.

(b) The adjoint bilinear form $B^* : H_0^1(U) \times H_0^1(U) \longrightarrow \mathbb{R}$ *is defined by*

$$B^*[v, u] := B[u, v]$$

for all $u, v \in H_0^1(U)$.

(c) We say that $v \in H_0^1(U)$ *is a weak solution of the adjoint problem*

$$\begin{cases} L^*v = f & \text{in } U, \\ \quad v = 0 & \text{on } \partial U, \end{cases}$$

provided that

$$B^*[v, u] = (f, u)$$

for all $u \in H_0^1(U)$.

Theorem 2.7 (Second Existence Theorem for weak solutions)**.** *There holds the following.*

(a) Precisely one of the following statements holds:

(α) For each $f \in L^2(U)$ *there exists a unique weak solution* u *of the boundary value problem*

$$\begin{cases} Lu = f & \text{in } U, \\ \quad u = 0 & \text{on } \partial U, \end{cases} \tag{2.4}$$

or else

(β) there exists a weak solution $u \not\equiv 0$ of the homogeneous problem

$$\begin{cases} Lu = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.5}$$

(b) *Furthermore, should assertion (β) hold, the dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of* (2.5) *is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of*

$$\begin{cases} L^*v = 0 & \text{in } U, \\ v = 0 & \text{on } \partial U. \end{cases} \tag{2.6}$$

(c) *Finally, the boundary value problem* (2.4) *has a weak solution if and only if*

$$(f, v) = 0 \ \text{ for all } \ v \in N^*.$$

Proof. **Step 1:** As in the proof of Theorem 2.4, choose $\mu = \gamma$ and define the bilinear form

$$B_\gamma[u, v] := B[u, v] + \gamma(u, v),$$

corresponding to the operator $L_\gamma u := Lu + \gamma u$. Thus, for each $g \in L^2(U)$, there exists a unique $u \in H_0^1(U)$ solving

$$B_\gamma[u, v] = (g, v) \ \text{ for all } \ v \in H_0^1(U). \tag{2.7}$$

Write $u = L_\gamma^{-1} g$ whenever (2.7) holds.
Step 2: Observe that $u \in H_0^1(U)$ is a weak solution of (2.4) if and only if

$$B_\gamma[u, v] = (\gamma u + f, v) \ \text{ for all } \ v \in H_0^1(U), \tag{2.8}$$

that is, if and only if

$$u = L_\gamma^{-1}(\gamma u + f). \tag{2.9}$$

We can rewrite this as

$$u - Ku = h, \tag{2.10}$$

where $Ku := \gamma L_\gamma^{-1} u$ and $h := L_\gamma^{-1} f$.
Step 3: We now claim that $K : L^2(U) \longrightarrow L^2(U)$ is a bounded, linear, compact operator. Indeed, from our choice of γ and the energy estimates from the previous section, we note that if (2.7) holds, then

$$\beta \|u\|_{H_0^1(U)}^2 \leq B_\gamma[u, u] = (g, u) \leq \|g\|_{L^2(U)} \|u\|_{L^2(U)} \leq \|g\|_{L^2(U)} \|u\|_{H_0^1(U)},$$

and so

$$\|Kg\|_{L^2(U)} \leq \|Kg\|_{H_0^1(U)} = \|\gamma L_\gamma^{-1} g\|_{H_0^1(U)} = \|u\|_{H_0^1(U)} \leq C\|g\|_{L^2(U)} \text{ for } g \in L^2(U)$$

for some suitable constant $C > 0$. However, since $H_0^1(U) \subset\subset L^2(U)$ by the Rellich-Kondrachov compactness theorem (see Theorem A.18), we conclude that K is a compact operator.

Step 4: By the Fredholm alternative, we conclude either

(α) for each $h \in L^2(U)$ the equation $u - Ku = h$ has a unique solution $u \in L^2(U)$; or else

(β) the equation $u - Ku = 0$ has non-trivial solutions in $L^2(U)$.

Should assertion (α) hold, then according to (2.8)–(2.10), there exists a unique weak solution of problem (2.4). On the other hand, should assertion (β) be valid, then necessarily $\gamma \neq 0$ and we recall that the dimension of the space N of the solutions of $u - Ku = 0$ is finite and equals the dimension of the space N^* of solutions of the equation

$$v - K^* v = 0. \tag{2.11}$$

However, we have that (β) holds if and only if u is a weak solution of (2.5) and that (2.11) holds if and only if v is a weak solution of (2.6).

Step 5: Finally, we recall equation $u - Ku = h$ in (α) has a solution if and only if

$$(h, v) = 0$$

for all v solving (2.11). However, from (2.11) we compute that

$$(h, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^* v) = \frac{1}{\gamma}(f, v).$$

Hence, the boundary value problem (2.4) has a solution if and only if $(f, v) = 0$ for all weak solutions v of (2.6).

□

Definition 2.3. *We say $\lambda \in \Sigma$, the (real) spectrum of the operator L, if the boundary value problem*

$$\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

has a non-trivial solution w, in which case λ is called an **eigenvalue** *of L, w a corresponding* **eigenfunction**. *Particularly, the partial differential equation $Lu = \lambda u$ for $L = -\Delta$ is often called the Helmholtz equation.*

Theorem 2.8 (Third Existence Theorem for weak solutions)**.** *There holds the following.*

(a) There exists an at most coutable set $\Sigma \subset \mathbb{R}$ such that the boundary value problem

$$\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.12}$$

has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

(b) If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$, the values of a non-decreasing sequence with $\lambda_k \longrightarrow \infty$.

Proof. **Step 1:** Let γ be the constant from Theorem 2.3 and assume $\lambda > -\gamma$. Without loss of generality, we also assume $\gamma > 0$.

According to the Fredholm alternative, problem (2.12) has a unique weak solution for each $f \in L^2(U)$ if and only if $u \equiv 0$ is the only weak solution of the homogeneous problem

$$\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

This is in turn true if and only if $u \equiv 0$ is the only weak solution of

$$\begin{cases} Lu + \gamma u = (\gamma + \lambda) u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.13}$$

Now (2.13) holds precisely when

$$u = L_\gamma^{-1}(\gamma + \lambda) u = \frac{\gamma + \lambda}{\gamma} K u, \tag{2.14}$$

where, as in the proof of the previous theorem, $Ku := \gamma L_\gamma^{-1} u$ and K is a bounded and compact linear operator on $L^2(U)$.

Now, if $u \equiv 0$ is the only solution of (2.14), we see

$$\frac{\gamma}{\gamma + \lambda} \text{ is not an eigenvalue of } K. \tag{2.15}$$

Hence, we see that (2.12) has a unique weak solution for each $f \in L^2(U)$ if and only if (2.15) holds.

Step 2: According to Theorem 2.6, the set of all non-zero eigenvalues of K forms either finite set or else the values of a sequence converging to zero. In the second case, $\lambda > -\gamma$ and (2.14) imply that (2.12) has a unique weak solution for all $f \in L^2(U)$ except for a sequence $\lambda_k \longrightarrow \infty$. □

Theorem 2.9 (Boundedness of the inverse)**.** *If $\lambda \notin \Sigma$, there exists a positive constant C such that*

$$\|u\|_{L^2(U)} \le C \|f\|_{L^2(U)},$$

whenever $f \in L^2(U)$ and $u \in H_0^1(U)$ is the unique weak solution of

$$\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

The constant C depends only on λ, U, and the coefficients of the elliptic operator L.

2.3 Eigenvalues and Eigenfunctions

This section is somewhat of a digression from the rest of the chapter in that we study eigenvalues for symmetric uniformly elliptic operators. We feel that this follows naturally from the previous section as we continue to examine properties of compact operators in the setting of partial differential equations. As such, we only consider symmetric elliptic operators, but the theory certainly extends to the non-symmetric setting (see [8]).

We consider the boundary value problem

$$\begin{cases} Lw = \lambda w & \text{in } U, \\ w = 0 & \text{on } \partial U, \end{cases} \tag{2.16}$$

where $U \subset \mathbb{R}^n$ is open, bounded and connected. We say $\lambda \in \mathbb{C}$ is an eigenvalue of L provided there exists a non-trivial solution w of problem (2.16) where w is called the corresponding eigenfunction of λ. As we shall see, L is a compact and symmetric linear operator (actually it is really the inverse operator L^{-1} that satisfies these properties) and therefore, elementary spectral theory indicates the spectrum Σ of L is positive, real and at most countable. In particular, we take L to be of the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j},$$

where $a^{ij} \in C^\infty(\bar{U})$ and $a^{ij} = a^{ji}$ for $i, j = 1, 2, \ldots, n$. We note that the associated bilinear form $B[\cdot, \cdot]$ associated with this eigenvalue problem is symmetric, i.e., $B[u, v] = B[v, u]$ for all $u, v \in H_0^1(U)$ since L is formally symmetric.

Theorem 2.10 (Eigenvalues of symmetric elliptic operators)**.** *There hold the following.*

(a) Each eigenvalue of L is real.

(b) Furthermore, if we repeat each eigenvalue according to its finite multiplicity, we have

$$\Sigma = \{\lambda_k\}_{k=1}^\infty$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots.$$

and $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$.

(c) Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^n$ of $L^2(U)$ where $w_k \in H_0^1(U)$ is an eigenvalue corresponding to λ_k in (2.16).

Remark 2.3. *The first eigenvalue $\lambda_1 > 0$ is often called the* **principal eigenvalue** *of L. Moreover, as examined in the next chapter, basic regularity theory ensures the eigenfunctions w_k, for $k = 1, 2, \ldots,$ actually belong to $C^\infty(U)$. In fact, they belong to $C^\infty(\bar{U})$ provided that the boundary ∂U is smooth.*

Proof. In fact, it is simple to show that $S = L^{-1}$ is a bounded and compact linear operator on $L^2(U)$. More precisely, for $f \in L^2(U)$, $Sf = u$ means $u \in H_0^1(U)$ is the weak solution of

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

Now, we claim that S is also symmetric. To see this, let $f, g \in L^2(U)$ and take $Sf = u$ and $Sg = v$. Notice that

$$(Sf, g) = (u, g) = B[u, v]$$

and

$$(f, Sg) = (f, v) = B[u, v].$$

Hence, the basic theory of compact, symmetric linear operators on Hilbert spaces imply the eigenvalues of S are real, positive and its corresponding eigenfunctions make up an orthonormal basis of $L^2(U)$. Moreover, for $\eta \neq 0$ and $\lambda = \eta^{-1}$, there holds $Sw = \eta w$ if and only if $Lw = \lambda w$. Thus, the same properties translate to the eigenvalues and eigenfunctions of L as well. This completes the proof.

□

Theorem 2.11 (Variational principle for the principal eigenvalue)**.** *There hold the following statements.*

(a) Rayleigh's formula holds, i.e.,

$$\lambda_1 = \min_{\|u\|_{L^2(U)}} \{B[u, v] \mid u \in H_0^1(U)\} = \min_{u \neq 0 \text{ in } H_0^1(U)} \frac{B[u, u]}{\|u\|^2_{L^2(U)}}.$$

(b) Furthermore, the above minimum is attained by a function $w_1 \in H_0^1(U)$, positive within U, which is also a weak solution of

$$\begin{cases} Lu = \lambda_1 u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

(c) The principle eigenvalue is simple, i.e., if $u \in H_0^1(U)$ is any weak solution of

$$\begin{cases} Lu = \lambda_1 u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

then u is a multiple of w_1. Therefore, the eigenvalues of L can be ordered as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots.$$

2.4 Topological Fixed Point Theorems

This section introduces topological fixed point theorems from functional analysis to establish the existence of weak solutions to a class of nonlinear elliptic PDEs.

2.4.1 Brouwer's Fixed Point Theorem

Before stating and proving Schauder's fixed point theorem, we state and prove Brouwer's fixed point theorem, since we will need it to prove Schauder's version. In particular, Schauder's theorem will be a generalization of Brouwer's to infinite dimensional Banach spaces. We adopt the notation that $B_r(x)$ or $B(x,r) \subset \mathbb{R}^n$ to represent the ball of radius r with center $x \in \mathbb{R}^n$, and we denote its closure by $\bar{B}_r(x)$ or $\bar{B}(x,r)$, respectively.

Theorem 2.12 (Brouwer's Fixed Point Theorem)**.** *Assume $u : \bar{B}_1(0) \to \bar{B}_1(0)$ is continuous. Then u has a fixed point, that is, there exists a point $x \in \bar{B}_1(0))$ with $u(x) = x$.*

To prove this, we exploit the fact that the unit sphere is not a retract of the closed unit ball. Namely, we prove

Theorem 2.13 (No Retraction Theorem)**.** *There is no continuous function*

$$u : \bar{B}_1(0) \longrightarrow \partial B_1(0)$$

such that $u \equiv Identity$ on $\partial B_1(0)$.

Proof. We proceed with a topological degree argument (see Chapter 1 in [19]). Assume that the unit sphere is a retract of the closed unit ball and a retraction mapping is given by u. Then, homotopy invariance ensures that $deg(u, B_1(0), 0) = deg(Identity, B_1(0), 0) = 1$ and thus there exists an interior point $x \in B_1(0)$ such that $u(x) = 0$. This is a contradiction with the assumption that $u(\bar{B}_1(0)) \subseteq \partial B_1(0)$. □

Proof of Brouwer's Fixed Point Theorem. Assume that $u(x) \neq x$ for all $x \in \bar{B}_1(0)$. Thus, we can define a map $w : \bar{B}_1(0) \longrightarrow \partial B_1(0)$ by letting w be the intersection of $\partial B_1(0)$ with the straight line starting at $u(x)$ and passing through x and ending on the boundary. This terminal boundary point is equal to $w(x)$, or more precisely,

$$w(x) = x + \gamma(u(x) - x),$$

where $\gamma = \gamma(x)$ is a real-valued map that ensures that $w(x)$ has unit norm. Clearly, w is continuous and $w(x) = x$ for all $x \in \partial B_1(0)$. Therefore, this implies that the unit sphere is a retract of the closed unit ball and we arrive at a contradiction with Theorem 2.13. This completes the proof of the theorem. □

Remark 2.4. *Brouwer's fixed point theorem generalizes to bounded and closed convex subsets in $\mathbb{R}^n$, since such proper subsets with non-empty interior are homeomorphic to the closed unit ball.*

2.4.2 Schauder's Fixed Point Theorem

Let us consider a Banach space X with norm $\|\cdot\|$.

Theorem 2.14 (Schauder)**.** *Suppose that K is a compact and convex subset of X. Assume that $A : K \to K$ is continuous. Then A has a fixed point in K.*

Proof. **Step 1:** Fix $\epsilon > 0$. Since K is compact, we can choose finitely many points $u_1, u_2, ..., u_{N_\epsilon}$ so that the collection of open balls $\{B(u_i, \epsilon)\}_{i=1}^{N_\epsilon}$ is a cover for K, i.e., $K \subset \bigcup_{i=1}^{N_\epsilon} B(u_i, \epsilon)$. Now let K_ϵ be the closed convex hull of the points $\{u_1, u_2, ..., u_{N_\epsilon}\}$:

$$K_\epsilon = \Big\{ \textstyle\sum_{i=1}^{N_\epsilon} \lambda_i u_i \,|\, 0 \leq \lambda_i \leq 1, \sum_{i=1}^{N_\epsilon} \lambda_i = 1 \Big\}.$$

So $K_\epsilon \subset K$ from the convexity of K and by definition of K_ϵ.

Let us define the operator $P_\epsilon : K \longrightarrow K$ by

$$P_\epsilon(u) = \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))u_i}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))} \text{ for } u \in K.$$

Remark 2.5. *We define the distance of $x \in X$ from a subset $Y \subset X$ by*

$$dist(x, Y) = \inf_{y \in Y} dist(x, y) = \inf_{y \in Y} \|x - y\|.$$

$P_\epsilon : K \longrightarrow K$ is well-defined since the denominator $\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))$ is never zero since $K \subset \bigcup_{i=1}^{N_\epsilon} B(u_i, \epsilon)$, i.e., u belongs to at least one of the open balls in the cover.

Step 2: In addition, $P_\epsilon : K \longrightarrow K$ is continuous. Suppose $\{v_k\} \longrightarrow v$ in K. Define for each $j = 1, ..., N_\epsilon$ the operator $P_\epsilon^j : K \longrightarrow K$ by

$$P_\epsilon^j(u) = \frac{dist(u, K - B(u_j, \epsilon))u_j}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i, \epsilon))} \text{ for } u \in K.$$

Then for some constant M,

$$\begin{aligned} \|P_\epsilon^j(v_k) - P_\epsilon^j(v)\| &\leq M \cdot \inf_{y \in K - B(u_j)} |\|v_k - y\| - \|v - y\|| \\ &\leq M \cdot \inf_{y \in K - B(u_j)} \|v_k - v\| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned}$$

Hence, each P_ϵ^j is continuous so therefore

$$P_\epsilon = \sum_{j=1}^{N_\epsilon} P_\epsilon^j$$

is continuous. Moreover, for $u \in K$ we have

$$
\begin{aligned}
\|P_\epsilon(u) - u\| &= \Big\| \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i,\epsilon)) u_i}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i,\epsilon))} - u \Big\| \le \Big\| \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i,\epsilon))(u_i - u)}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i,\epsilon))} \Big\| \\
&\le \frac{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i,\epsilon)) \|u_i - u\|}{\sum_{i=1}^{N_\epsilon} dist(u, K - B(u_i,\epsilon))} \le \|u_i - u\| \le \epsilon.
\end{aligned}
$$

Step 3: Now consider the operator $A_\epsilon : K_\epsilon \longrightarrow K_\epsilon$ defined by $A_\epsilon[u] := P_\epsilon[A(u)]$, $(u \in K_\epsilon)$. As remarked earlier, we note that K_ϵ is homeomorphic to the closed unit ball $\bar{B}(0,1)$ in the Euclidean space $\mathbb{R}^{M_\epsilon}$ for some $M_\epsilon \le N_\epsilon$. With this result, we can apply Brouwer's fixed point theorem to obtain the existence of a fixed point $u_\epsilon \in K_\epsilon$ with $A_\epsilon[u_\epsilon] = u_\epsilon$.

Step 4: We have that $\{u_\epsilon\}_{\epsilon>0}$ forms a sequence in K. The compactness of K implies that there is a subsequence, $\{u_{\epsilon_j}\}_{\epsilon_j>0}$, of $\{u_\epsilon\}_{\epsilon>0}$ that converges to some element $v \in K$. We now will show that this element v is in fact a fixed point of A. Using the bound from Step 2, one can establish that

$$
\|u_{\epsilon_j} - A[u_{\epsilon_j}]\| = \|A_{\epsilon_j}[u_{\epsilon_j}] - A[u_{\epsilon_j}]\| = \|P_{\epsilon_j}[A[u_{\epsilon_j}]] - A[u_{\epsilon_j}]\| \le \epsilon_j.
$$

By utilizing the continuity of A, as $\epsilon_j \longrightarrow 0$ then the bound gives us that $\|v - Av\| \le 0$ and thus $Av - v = 0$. □

2.4.3 Schaefer's Fixed Point Theorem

We shall deduce Schaefer's fixed point theorem from Schauder's. We shall see that this theorem is much more useful in application to PDEs since we work with compact operators rather than compact subsets of our Banach space X. However, before proceeding, we give two equivalent definitions on the notion of a compact operator or map.

Definition 2.4. *A (nonlinear) mapping $A : X \longrightarrow X$ on a Banach space X is compact if*

1. *for each bounded sequence $\{u_k\}_{k=1}^\infty$ in X, the sequence $\{A[u_k]\}_{k=1}^\infty$ is precompact, i.e., has a convergent subsequence in X.*

2. *for each bounded set $B \subset X$, $A(B)$ is precompact in X, i.e., its closure in X is a compact subset of X.*

Remark 2.6. *The former definition of sequential compactness was already provided in the previous section concerning the Fredholm alternative.*

Theorem 2.15 (Schaefer)**.** *Suppose $A : X \longrightarrow X$ is a continuous and compact mapping. Assume further that the set $S = \{u \in X \mid u = \lambda A[u]$, for some $0 \le \lambda \le 1\}$ is bounded. Then A has a fixed point in X.*

Proof. Suppose $u = \lambda A[u]$ for some $\lambda \in [0,1]$. Since S is bounded, we can find $M > 0$ such that $\|u\| < M$. Define $\bar{A} : \bar{B}(0,M) \longrightarrow \bar{B}(0,M)$ by

$$\bar{A}[u] = \begin{cases} A[u] & \text{if } \|A[u]\| \le M, \\ \dfrac{M}{\|A[u]\|} A[u] & \text{if } \ \|A[u]\| \ge M. \end{cases} \tag{2.17}$$

Set K to be the closed convex hull of $\bar{A}(B(0,1))$. Since A is compact, and any scalar multiple of a compact operator is compact implies that $\bar{A}$ is compact as well. Using the result that the convex hull of a precompact set is precompact, we deduce that K is a convex, closed and precompact subset of X. Hence K is a compact and convex subset of X and $\bar{A} : K \longrightarrow K$ is a compact and continuous map. By Schauder's fixed point theorem, there exists a fixed point $u^* \in K$ with $\bar{A}[u^*] = u^*$.

We will now show that u^* is also a fixed point of A. Assume otherwise; so that $\|A[u^*]\| > 0$ and $u^* = \lambda A[u^*]$ with $\lambda = \frac{M}{\|A[u^*]\|} < 1$. However, $\|u^*\| = \|\bar{A}[u^*]\| = M$ since $\|\lambda A[u^*]\| = \frac{M\|A[u^*]\|}{\|A[u^*]\|} = \bar{A}[u^*] = M$, a contradiction.

□

2.4.4 Application to Nonlinear Elliptic Boundary Value Problems

We focus on solving a class of non-linear elliptic PDEs which can be treated as compact operators on some suitable function space. In such cases, Schaefer's fixed point theorem can be applied. We provide a fundamental example.

Consider the semilinear boundary-value problem

$$\begin{cases} -\Delta u + b(Du) + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.18}$$

where U is a bounded and open subset of $\mathbb{R}^n$ and ∂U is smooth, $b : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth and Lipschitz continuous so that

$$|b(p)| \le C(|p| + 1)$$

for some positive constant C. We will prove the following claim.

Theorem 2.16. *If $\mu > 0$ is sufficiently large, there exists a function $u \in H_0^1(U)$ solving the boundary-value problem (2.18). Furthermore, u also belongs to $H^2(U)$.*

Proof. We prove the theorem in three main steps.

Step 1: Given $u \in H_0^1(U)$, set $f := -b(Du)$. So by Lipschitz continuity we can show $f \in L^2(U)$ since

$$|f(u)| = |b(Du)| \le C(|Du| + 1),$$

then

$$\|f\|_{L^2(U)} \le \|Du\|_{L^2(U)} + C \le \|u\|_{H_0^1(U)} + C < \infty.$$

Now we will define the map $A : H_0^1(U) \to H_0^1(U)$. Formulate the linear boundary value problem

$$\begin{cases} -\Delta w + \mu w = f(u) & \text{in } U \\ w = 0 & \text{on } \partial U. \end{cases} \tag{2.19}$$

Since f was shown to belong to $L^2(U)$, linear PDE theory ensures the existence of a unique weak solution $w \in H_0^1(U)$ of the linear problem (2.19). Hence, for $u \in H_0^1(U)$, define $A[u] = w$. Moreover, basic elliptic regularity theory yields the estimate

$$\|w\|_{H^2(U)} = \|A[u]\|_{H^2(U)} \leq C\|f\|_{L^2(U)}$$

for some constant C (see Theorem 3.14 in the next chapter). Combining this with the above L^2 estimate on f, we get

$$\|w\|_{H^2(U)} = \|A[u]\|_{H^2(U)} \leq C(\|u\|_{H_0^1(U)} + 1)$$

for some constant C.

Step 2: We will show that $A : H_0^1(U) \to H_0^1(U)$ is a continuous and compact mapping. Suppose that $\{u_k\}_{k=1}^{\infty} \longrightarrow u$ in $H_0^1(U)$. Since

$$\|w\|_{H^2(U)} \leq C(\|u\|_{H_0^1(U)} + 1) \text{ for each } k \in \mathbb{N},$$

this implies that

$$\sup_k \|w_k\|_H^2(U) < \infty.$$

Then, as a consequence of the Rellich-Kondrachov compactness theorem (see Theorem A.18), there is a subsequence $\{w_{k_j}\}_{j=1}^{\infty}$ and a function $w \in H_0^1(U)$ with $\{w_{k_j}\}_{j=1}^{\infty} \longrightarrow w$ in $H_0^1(U)$. Note that each element of the subsequence satisfies $-\Delta w_{k_j} + \mu w_{k_j} = b(Du_{k_j})$. Now if we multiply this by any $v \in H_0^1(U)$ and integrate over U we obtain

$$\int_U -\Delta w_{k_j} v + \mu w_{k_j} v \, dx = -\int_U b(Du_{k_j}) v \, dx.$$

Integration by parts on the first term yields

$$\int_U Dw_{k_j} \cdot Dv + \mu w_{k_j} v \, dx = -\int_U b(Du_{k_j}) v \, dx.$$

Taking the limit as $j \longrightarrow \infty$ gives us

$$\int_U Dw \cdot Dv + \mu w v \, dx = -\int_U b(Du) v \, dx \text{ for all } v \in H_0^1(U).$$

This shows that $A[u] = w$ and $A[u_k] \longrightarrow A[u]$ in $H_0^1(U)$ given $u_k \longrightarrow u$ in $H_0^1(U)$. So A is a continuous map.

It is similar to show that A is compact. Take $\{u_k\}_{k=1}^\infty$ to be a bounded sequence in $H_0^1(U)$ We have already shown that $\sup_k \|w_k\|_{H^2(U)} < \infty$ so $\{A[u_k]\}_{k=1}^\infty$ is a bounded sequence in $H^2(U) \cap H_0^1(U)$; therefore it must contain a strongly convergent subsequence in $H_0^1(U)$. Again, this is a consequence of the Rellich-Kondrachov compactness theorem, which says that $H^2(U)$ is compactly embedded into $H_0^1(U)$.

Step 3: The final part to show is that if μ is sufficiently large, the set

$$S = \left\{ u \in H_0^1(U) \,|\, u = \lambda A[u] \text{ for some } 0 < \lambda \le 1 \right\}$$

is a bounded set in $H_0^1(U)$. So let us assume $u \in S$ so that $u/\lambda = A[u]$ or $u \in H^2(U) \cap H_0^1(U)$ and $-\Delta u + \mu u = \lambda b(Du)$ a.e. in U. Multiply (2.18) by u then integrate over U to get

$$\begin{aligned}
\int_U (-\Delta + \mu u) u \, dx &= \int_U Du \cdot Du + \mu |u|^2 \, dx = \int_U |Du|^2 + \mu |u|^2 \, dx \\
&= -\int_U \lambda b(Du) u \, dx \le \int_U |b(Du)||u| \, dx \le \int_U C(|Du| + 1)|u| \, dx \\
&\le \frac{1}{2} \int_U (|Du| + 1)^2 + C|u|^2 \, dx \le \frac{1}{2} \int_U |Du|^2 + K \int_U |u|^2 + 1 \, dx
\end{aligned}$$

for some constants C and K independent of λ. This implies that

$$\frac{1}{2} \int_U |Du|^2 \, dx + (\mu - K) \int_U |u|^2 \, dx \le K \int_U dx =: \frac{1}{2} M^2$$

where M is a positive constant. From our bounds, note that M is independent of the choice of $u \in S$. So if we choose

$$\mu = K + \frac{1}{2}$$

then

$$\frac{1}{2} \int_U |u|^2 + |Du|^2 \, dx \le \frac{1}{2} M^2.$$

Hence, $\|u\|_{H_0^1(U)} \le M < \infty$ for all $u \in S$, i.e., S is bounded in $H_0^1(U)$.

Finally apply Schaefer's fixed point theorem on $X = H_0^1(U)$ to show that A has a fixed point in $H^2(U) \cap H_0^1(U)$. By our construction of the mapping A, this fixed point solves our semilinear elliptic problem.

$\square$

2.5 Perron Method

In this section, we introduce the Perron method to obtain the existence of classical solutions to Dirichlet problems on general domains provided that the solutions of the same problems on ball domains are known to exist. For simplicity and as our main example, we consider

Laplace's equation on general domains. That is, let U be a bounded domain in $\mathbb{R}^n$ and φ be a continuous function on ∂U. Consider

$$\begin{cases} -\Delta u = 0 & \text{in } U, \\ \quad\ u = \varphi & \text{on } \partial U. \end{cases} \tag{2.20}$$

Note that, if U is an open ball, then the solutions of (2.20) are given by Poisson's formula via the Green's function on a ball domain. Otherwise, we shall use the Perron method in which the maximum principle plays an important role. First, we define continuous subharmonic and superharmonic functions based on the maximum principle.

Definition 2.5. *Let U be a bounded domain in $\mathbb{R}^n$ and v be a continuous function in U. Then v is subharmonic (respectively superharmonic) in U if for any ball $B \subset U$ and any harmonic function $w \in C(\bar{B})$,*

$$v \leq (\textit{respectively} \geq)\, w \ \textit{ on } \ \partial B \ \textit{ implies } v \leq (\textit{respectively} \geq)\, w \ \textit{ in } \ B.$$

Before introducing the Perron method, we start with some preliminary results.

Lemma 2.1. *Let U be a bounded domain in $\mathbb{R}^n$ and $u, v \in C(\bar{U})$. Suppose u is subharmonic in U and v is superharmonic in U with $u \leq v$ on ∂U. Then $u \leq v$ in U.*

Proof. Without loss of generality, let us assume U is connected. Indeed, $u - v \leq 0$ on ∂U. Set $M = \max_{\bar{U}}(u - v)$ and

$$D = \{x \in U \,|\, u(x) - v(x) = M\} \subset U.$$

We claim that D is both an open and relatively closed subset of U and so, by the connectedness of U, either $D = \emptyset$ or $D = U$. It is clear that D is a relatively closed subset by the continuity of u and v. To show D is open, take any point $x_0 \in D$ and take $r < dist(x_0, \partial U)$. Let $\bar{u}$ and $\bar{v}$ solve, respectively,

$$\begin{aligned} \Delta\bar{u} = 0, \ \text{ in } \ B_r(x_0), \quad \bar{u} = u \ \text{ on } \ \partial B_r(x_0), \\ \Delta\bar{v} = 0, \ \text{ in } \ B_r(x_0), \quad \bar{v} = v \ \text{ on } \ \partial B_r(x_0). \end{aligned}$$

Now, the existence of the solutions $\bar{u}$ and $\bar{v}$ is guaranteed by Poisson's formula for $U = B_r(x_0)$. Moreover, by recalling the definitions of subsolutions and supersolutions, we deduce that $u \leq \bar{u}$ and $\bar{v} \leq v$ in $B_r(x_0)$. Therefore,

$$\bar{u} - \bar{v} \geq u - v \ \text{ in } \ B_r(x_0).$$

Next,

$$\begin{cases} \Delta(\bar{u} - \bar{v}) = 0 & \text{in } B_r(x_0), \\ \quad\ \bar{u} - \bar{v} = u - v & \text{on } \partial B_r(x_0). \end{cases}$$

With $u - v \le M$ on $\partial B_r(x_0)$, the maximum principle implies $\bar{u} - \bar{v} \le M$ in $B_r(x_0)$. In particular,

$$M \ge (\bar{u} - \bar{v})(x_0) \ge (u - v)(x_0) = M.$$

Hence, $(\bar{u} - \bar{v})(x_0) = M$ and then $\bar{u} - \bar{v}$ has an interior maximum at x_0. Then, by the strong maximum principle, $\bar{u} - \bar{v} \equiv M$ in $B_r(\bar{x}_0)$, i.e., $u - v = M$ on $\partial B_r(x_0)$, and this holds for all $r < dist(x_0, \partial U)$. Then $u - v = M$ in $B_r(x_0)$ and thus $B_r(x_0) \subset D$. We conclude that $D = \emptyset$ or $D = U$, i.e., either $u - v$ attains its maximum only at ∂U or $u - v$ is constant in U. By $u \le v$ in ∂U, we have $u \le v$ in U in both cases. $\square$

Remark 2.7. *In the proof above, we actually proved the strong maximum principle: Either $u < v$ in U or $u - v$ is constant in U.*

Lemma 2.2. *Let $v \in C(\bar{U})$ be a subharmonic function in U and $B \subset\subset U$ is a ball. Let w be defined by $w = v$ in $\bar{U} \backslash B$ and $\Delta w = 0$ in B. Then w is a subharmonic function in U and $v \le w$ in $\bar{U}$.*

Remark 2.8. *Here, the function w is often called the* **harmonic lifting** *of v in B.*

Proof of Lemma 2.2. The existence of w is implied by Poisson's formula for $U = B$. Also, w is smooth in B and continuous in $\bar{U}$. We also have $v \le w$ in B by definition of subharmonic functions in U. Now take any $B' \subset\subset U$ and consider a harmonic function $u \in C(\bar{B}')$ with $w \le u$ on $\partial B'$. By $v \le w$ on $\partial B'$, we have $v \le u$ on $\partial B'$. Then, v is subharmonic and u is harmonic in B' with $v \le u$ on $\partial B'$. By Lemma 2.1, we have $v \le u$ in B'. Hence, $w \le u$ in $B \backslash B'$. Additionally, both w and u are harmonic in $B \cap B'$ and $w \le u$ on $\partial(B \cap B')$. So by the maximum principle, we have $w \le u$ in $B \cap B'$. Hence, $w \le u$ in B'. We then conclude that, by definition, w is subharmonic in U. This completes the proof of the lemma. $\square$

Now we are ready to solve (2.20) via the Perron method. Set

$$u_\varphi(x) = \sup\{v(x) \,|\, v \in C(\bar{U}) \text{ is subharmonic in } U,\, v \le \varphi \text{ on } \partial U\}. \tag{2.21}$$

Ultimately, our goal is to show that this function u_φ is indeed a solution of the Dirichlet problem (2.20). The first step in the Perron method is to show that u_φ in (2.21) is indeed harmonic in U.

Lemma 2.3. *Let U be a bounded domain in $\mathbb{R}^n$ and φ be a continuous function on ∂U. Then u_φ defined in (2.21) is harmonic in U.*

Proof. Set

$$\mathcal{S}_\varphi = \{v \,|\, v \in C(\bar{U}) \text{ is subharmonic in } U,\, v \le \varphi \text{ on } \partial U\},$$

and we set $\mathcal{S} = \mathcal{S}_\varphi$ if there is no confusion in its meaning. Then for any $x \in U$,

$$u_\varphi(x) = \sup\{v(x) \,|\, v \in \mathcal{S}\}.$$

Step 1: The quanitity u_φ is well defined.

To show this, first set

$$m = \min_{\partial U} \varphi \text{ and } M = \max_{\partial U} \varphi.$$

We note that the constant function m is in $\mathcal{S}$ and thus the set $\mathcal{S}$ is non-empty. Next, the constant function M is clearly harmonic in U with $\varphi \leq M$ on ∂U. By Lemma 2.1, for any $v \in \mathcal{S}$,

$$v \leq M \text{ in } \bar{U}.$$

Thus u_φ is well-defined and $u_\varphi \leq M$ in U.

Step 2: We show $\mathcal{S}$ is closed by taking the maximum among finitely many functions in $\mathcal{S}$.

Choose arbitrary $v_1, v_2, \ldots, v_k \in \mathcal{S}$ and set

$$v = \max\{v_1, v_2, \ldots, v_k\}.$$

It follows easily, by definition, that v is subharmonic in U. Hence, $v \in \mathcal{S}$.

Step 3: We prove that u_φ is harmonic in any $B_r(x_0) \subset U$.

By definition of u_φ, there exists a sequence of functions $v_i \in \mathcal{S}$ such that

$$\lim_{i\to\infty} v_i(x_0) = u_\varphi(x_0).$$

We may replace v_i above by any $\tilde{v}_i \in \mathcal{S}$ with $\tilde{v}_i \geq v_i$ since

$$v_i(x_0) \leq \tilde{v}_i(x_0) \leq u_\varphi(x_0).$$

Replacing, if necessary, v_i by $\max\{m, v_i\} \in \mathcal{S}$, we may also assume

$$m \leq v_i \leq u_\varphi \text{ in } U.$$

For fix $B_r(x_0)$ and each v_i, we let w_i be the harmonic lifting in Lemma 2.2. Then $w_i = v_i$ in $U \backslash B_r(x_0)$ and

$$\begin{cases} \Delta w_i = 0 & \text{in } B_r(x_0), \\ w_i = v_i & \text{on } \partial B_r(x_0). \end{cases}$$

By Lemma 2.2, $w_i \in \mathcal{S}$ and $v_i \leq w_i$ in U. Moreover, w_i is harmonic in $B_r(x_0)$ and satisfies

$$\lim_{i\to\infty} w_i(x_0) = u_\varphi(x_0),$$
$$m \leq w_i \leq u_\varphi \text{ in } U,$$

for any $i = 1, 2, \ldots$. By the compactness of bounded harmonic functions (see Corollary 1.5), there exists a harmonic function w in $B_r(x_0)$ such that a subsequence of $\{w_i\}$, we still denote by $\{w_i\}$, converges to w on compact subsets of $B_r(x_0)$. We deduce that

$$w \leq u_\varphi \text{ in } B_r(x_0) \text{ and } w(x_0) = u_\varphi(x_0).$$

We now claim that $u_\varphi = w$ in $B_r(x_0)$. To see this, take any $\bar{x} \in B_r(x_0)$ and proceed similarly as before, with $\bar{x}$ replacing x_0. By definition of u_φ, there exists a sequence $\{\bar{v}_i\} \subset \mathcal{S}$ such that

$$\lim_{i\to\infty} \bar{v}_i(\bar{x}) = u_\varphi(\bar{x}).$$

As before, we can replace, if necessary, $\bar{v}_i$ by $\max\{\bar{v}_i, w_i\} \in \mathcal{S}$. So we may also assume that

$$w_i \le \bar{v}_i \le u_\varphi \text{ in } U.$$

For the fixed $B_r(x_0)$ and each $\bar{v}_i$, we let $\bar{w}_i$ be the harmonic lifting in Lemma 2.2. Then, $\bar{w}_i \in \mathcal{S}$ and $\bar{v}_i \le \bar{w}_i$ in U. Moreover, $\bar{w}_i$ is harmonic in $B_r(x_0)$ and satisfies

$$\lim_{i\to\infty} \bar{w}_i(\bar{x}) = u_\varphi(\bar{x}),$$
$$m \le \max\{\bar{v}_i, w_i\} \le \bar{w}_i \le u_\varphi \text{ in } U,$$

for any $i = 1, 2, \ldots$. Again, by compactness, there exists a harmonic function $\bar{w}$ in $B_r(x_0)$ with a maximum attained at x_0. Then, by the strong maximum principle applied to $w - \bar{w}$ in $B_{r'}(x_0)$ for any $r' < r$, we deduce that $w - \bar{w}$ is constant and thus is equal to zero. This implies $w = \bar{w}$ in $B_r(x_0)$ and particularly, $w(\bar{x}) = \bar{w}(\bar{x}) = u_\varphi(\bar{x})$. Hence, $w = u_\varphi$ in $B_r(x_0)$ since $\bar{x}$ was chosen arbitrarily in $B_r(x_0)$. This proves u_φ is harmonic in $B_r(x_0)$. □

Observe carefully that u_φ as given in the previous lemma is only defined in U. To discuss the limits of $u_\varphi(x)$ as x approaches the boundary, we must make some additional assumptions on the boundary of U, ∂U.

Lemma 2.4. *Let φ be a continuous function on ∂U and u_φ be the function defined in* (2.21)*. For some $x_0 \in \partial U$, suppose $w_{x_0} \in C(\bar{U})$ is a subharmonic function in U such that*

$$w_{x_0}(x_0) = 0, \quad w_{x_0}(x) < 0 \text{ for any } x \in \partial U \backslash \{x_0\}, \tag{2.22}$$

then

$$\lim_{x\to x_0} u_\varphi(x) = \varphi(x_0).$$

Proof. As before, consider the set

$$\mathcal{S}_\varphi = \{v \,|\, v \in C(\bar{U}) \text{ is subharmonic in } U, v \le \varphi \text{ on } \partial U\}.$$

To simplify notation, we just write $w = w_{x_0}$ and set $M = \max_{\partial U} |\varphi|$. Let $\varepsilon > 0$ be arbitrary, and by the continuity of φ at x_0, there exists a $\delta > 0$ such that

$$|\varphi(x) - \varphi(x_0)| < \varepsilon \text{ for any } x \in \partial U \cap B_\delta(x_0).$$

We then choose K suitably large so that $-Kw(x) \ge 2M$ for any $x \in \partial U \backslash B_\delta(x_0)$. Thus,

$$|\varphi(x) - \varphi(x_0)| < \varepsilon - Kw \text{ for } x \in \partial U.$$

Since $\varphi(x_0) - \varepsilon + Kw(x)$ is a subharmonic function in U with $\varphi(x_0) - \varepsilon + Kw \leq \varphi$ on ∂U, we have that $\varphi(x_0) - \varepsilon + Kw \in \mathcal{S}_\varphi$. The definition of u_φ then implies that

$$\varphi(x_0) - \varepsilon + Kw \leq u_\varphi \text{ in } U. \tag{2.23}$$

However, $\varphi(x_0) + \varepsilon - Kw$ is super-harmonic in U with $\varphi(x_0) + \varepsilon - Kw \geq \varphi$ on ∂U. Thus, for any $v \in \mathcal{S}_\varphi$, we obtain from Lemma 2.1

$$v(x) \leq \varphi(x_0) + \varepsilon - Kw(x) \text{ for } x \in U.$$

Again, by the definition of u_φ,

$$u_\varphi(x) \leq \varphi(x_0) + \varepsilon - Kw(x) \text{ for } x \in U. \tag{2.24}$$

Hence, (2.23) and (2.24) imply

$$|u_\varphi(x) - \varphi(x_0)| < \varepsilon - Kw(x) \text{ for } x \in U,$$

and since w is continuous so that $w(x) \longrightarrow w(x_0) = 0$ as $x \longrightarrow x_0$, we arrive at

$$\limsup_{x \to x_0} |u_\varphi(x) - \varphi(x_0)| < \varepsilon.$$

The desired result follows once after sending $\varepsilon \longrightarrow 0$.

$\square$

Remark 2.9. *The function w_{x_0} satisfying* (2.22) *is often called a barrier function. Barrier functions can be constructed for a large class of domains. One type of domain, for instance, is when U satisfies an* **exterior sphere condition** *at $x_0 \in \partial U$, i.e., there exists a ball $B_{r_0}(y_0)$ such that*

$$U \cap B_{r_0}(y_0) = \emptyset, \quad \bar{U} \cap \bar{B}_{r_0}(y_0) = \{x_0\}.$$

To construct a barrier function at x_0, we take

$$w_{x_0}(x) = \Gamma(x - y_0) - \Gamma(x_0 - y_0) \text{ for any } x \in \bar{U}$$

where Γ is the fundamental solution of Laplace's equation. Therefore, w_{x_0} is harmonic in U and satisfies (2.22). *In addition, we mention that the exterior sphere condition always holds for C^2 domains.*

Combining the previous lemmas and remark, we have essentially constructed a solution $u = u_\varphi$ to the Dirichlet problem (2.20). That is, we have shown the following existence result.

Theorem 2.17. *Let U be a bounded domain in $\mathbb{R}^n$ satisfying the exterior sphere condition at every boundary point. Then, for any $\varphi \in C(\partial U)$, the Dirichlet problem* (2.20) *admits a solution $u \in C^\infty(U) \cap C(\bar{U})$.*

In summary, the solvability of the Dirichlet problem for Laplace's equation depends on both the data g and the geometry of the domain U. As indicated in Lemma 2.4, the issue resolves around the following question. When can the harmonic function from the Perron method be extended continuously up to the boundary? In other words, when are the points of the boundary regular with respect to the Laplacian? Of course, g being continuous on ∂U and U satisfying the exterior sphere condition are enough to give a positive answer to this question. Alternatively, another criterion indicating when a boundary point is regular with respect to the Laplacian can be given in terms of 2-capacities. This criterion is called the Wiener criterion, and it easily generalizes to uniformly elliptic equations in divergence form.

Let $n \geq 3$ and

$$K^p = \{f : \mathbb{R}^n \to \mathbb{R}_+ \,|\, f \in L^{p^*}(\mathbb{R}^n), Df \in L^p(\mathbb{R}^n; \mathbb{R}^n)\}.$$

If $A \subset \mathbb{R}^n$, we define the p-capacity of A by

$$Cap_p(A) = \inf \Big\{ \int_{\mathbb{R}^n} |Df|^p \, dx \,:\, f \in K^p, A \subset interior\{f \geq 1\} \Big\}.$$

By regularization, note that

$$Cap_p(K) = \inf \Big\{ \int_{\mathbb{R}^n} |Df|^p \, dx \,:\, f \in C_c^\infty(\mathbb{R}^n),\ f \geq \chi_K \Big\}$$

for each compact set $K \subset \mathbb{R}^n$.

Let x_0 be a boundary point in ∂U. Then for any fixed $\lambda \in (0,1)$, let

$$A_j = \{x \notin U \,:\, |x - x_0| \leq \lambda^j\}.$$

The **Wiener criterion** states that x_0 is a regular boundary point of U if and only if the series

$$\sum_{j=0}^{\infty} \frac{Cap_2(A_j)}{\lambda^{j(n-2)}}$$

diverges.

2.6 Continuity Method

In this section, we introduce the continuity method to prove the existence of classical solutions to general uniformly elliptic equations of second-order. One crucial ingredient of the method relies on global $C^{2,\alpha}$ a priori estimates of solutions (see the Schauder estimates in Section 3.5) and this provides one important application of the regularity theory for such equations. In the next chapter, we will investigate the various types of regularity properties of solutions to uniformly elliptic equations in great detail.

Let $U \subset \mathbb{R}^n$ be a bounded domain, let a^{ij}, b^i and c be defined in U with a^{ij} symmetric. Consider the second-order elliptic operator

$$Lu = -a^{ij}(x)D_{ij}u + b^i(x)D_i u + c(x)u \text{ in } U$$

and assume L is uniformly elliptic in the following sense:

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for any } x \in U \text{ and } \xi \in \mathbb{R}^n$$

for some positive constant $\lambda > 0$.

We prove the following general existence result for solutions of Dirichlet boundary value problem with $C^{2,\alpha}$ boundary values involving the operator L with C^α coefficients.

Theorem 2.18. *Let $U \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, let L be a uniformly elliptic operator as defined as above with $c \geq 0$ in U and $a^{ij}, b, c \in C^\alpha(U)$ for some $\alpha \in (0,1)$. Then for any $f \in C^\alpha(\bar{U})$ and $\varphi \in C^{2,\alpha}(\bar{U})$, there exists a unique solution $u \in C^{2,\alpha}(\bar{U})$ of the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } U, \\ u = \varphi & \text{on } \partial U. \end{cases} \tag{2.25}$$

In fact, we shall prove the solvability of the boundary value problem (2.25) if the same is true for the boundary value problem with $L = -\Delta$, i.e., for Poisson's equation. Of course, the latter is a basic known result and so Theorem 2.18 follows accordingly.

Theorem 2.19. *Let $U \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain, let L be a uniformly elliptic operator as defined above with $c \geq 0$ in U and $a^{ij}, b, c \in C^\alpha(U)$ for some $\alpha \in (0,1)$. If the Dirichlet problem for Poisson's equation*

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = \varphi & \text{on } \partial U, \end{cases} \tag{2.26}$$

has a $C^{2,\alpha}(\bar{U})$ solution for all $f \in C^\alpha(\bar{U})$ and $\varphi \in C^{2,\alpha}(\bar{U})$, then the Dirichlet problem,

$$\begin{cases} Lu = f & \text{in } U, \\ u = \varphi & \text{on } \partial U, \end{cases} \tag{2.27}$$

also has a (unique) $C^{2,\alpha}(\bar{U})$ solution for all such f and φ.

Proof. Without loss of generality, we assume $\varphi \equiv 0$; otherwise, we consider $Lv = f - L\varphi$ in U and $v = 0$ on ∂U.

Consider the family of equations:

$$L_t u \equiv tLu + (1-t)(-\Delta)u = f$$

for $t \in [0,1]$. We note that $L_0 = -\Delta$ and $L_1 = L$.

If we write

$$L_t u = a_t^{ij}(x) D_{ij} u + b_t^i(x) D_i u + c_t(x) u,$$

we can easily verify that

$$a_t^{ij}(x)\xi_i\xi_j \geq \min(1, \lambda)|\xi|^2$$

for any $x \in U$ and $\xi \in \mathbb{R}^n$ and that

$$|a_t^{ij}|_{C^\alpha(\bar{U})}, |b_t^i|_{C^\alpha(\bar{U})}, |c_t|_{C^\alpha(\bar{U})} \leq \max(1, \Lambda)$$

independently of $t \in [0, 1]$. Thus,

$$|L_t u|_{C^\alpha(\bar{U})} \leq C|u|_{C^{2,\alpha}(U)}$$

where C is a positive constant depending only on $n, \alpha, \lambda, \Lambda$ and U. Then for each $t \in [0, 1]$, $L_t : X \to C^\alpha(U)$ is a bounded operator, where

$$X = \{u \in C^{2,\alpha}(\bar{U}) \,|\, u = 0 \ \text{ on } \ \partial U\}$$

is the Banach space equipped with the norm $|\cdot|_{C^{2,\alpha}(\bar{U})}$.

Define the set I containing the points $s \in [0, 1]$ such that the Dirichlet problem

$$\begin{cases} L_s u = f & \text{in } U, \\ \quad u = 0 & \text{on } \partial U, \end{cases} \tag{2.28}$$

is solvable in $C^{2,\alpha}(\bar{U})$ for any $f \in C^\alpha(\bar{U})$. We take an $s \in I$ and let $u = L_s^{-1} f$ be the (unique) solution. Then, standard global $C^{2,\alpha}$ estimates (cf. Theorem 3.18) and the maximum principle imply

$$|L_s^{-1} f|_{C^{2,\alpha}(U)} \leq C|f|_{C^\alpha(\bar{U})}.$$

For any $t \in [0, 1]$ and $f \in C^\alpha(\bar{U})$, we can write $L_t u = f$ as

$$L_s u = f + (L_s - L_t)u = f + (t - s)(\Delta u - Lu).$$

Hence, $u \in C^{2,\alpha}(\bar{U})$ is a solution of

$$\begin{cases} L_t u = f & \text{in } U, \\ \quad u = 0 & \text{on } \partial U, \end{cases}$$

if and only if

$$u = L_s^{-1}(f + (t - s)(\Delta - L)u).$$

For any $u \in X$, set

$$Tu = L_s^{-1}(f + (t - s)(\Delta u - Lu))$$

so that $T : X \longrightarrow X$ is an operator, and we claim T is a contraction mapping. Indeed, for any $u, v \in X$,

$$\begin{aligned}|Tu - Tv|_{C^{2,\alpha}(\bar{U})} &= |(t-s)L_s^{-1}((\Delta - L)(u-v))|_{C^{2,\alpha}(\bar{U})} \\ &\le C|t-s||(\Delta - L)(u-v)|_{C^{\alpha}(\bar{U})} \le C|t-s||u-v|_{C^{2,\alpha}(\bar{U})}.\end{aligned}$$

Therefore, $T : X \to X$ is a contraction mapping if $|t-s| < \delta := C^{-1}$. Hence, for any $t \in [0,1]$ with $|t-s| < \delta$, there exists a unique $u \in X$ such that $u = Tu$, i.e.,

$$u = L_s^{-1}(f + (t-s)(\Delta u - Lu)).$$

Namely, for any $t \in [0,1]$ with $|t-s| < \delta$ and any $f \in C^{\alpha}(\bar{U})$, there exists a solution of $u \in C^{2,\alpha}(\bar{U})$ of

$$\begin{cases} L_t u = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Therefore, if $s \in I$, then $t \in I$ for any $t \in [0,1]$ with $|t-s| < \delta$. So we can divide the interval $[0,1]$ into subintervals of length less than δ. By $0 \in I$, we deduce $1 \in I$. This completes the proof of the theorem. □

2.7 Calculus of Variations I: Minimizers and Weak Solutions

Another approach for establishing the existence of weak solutions to elliptic equations is through variational methods. This is especially important since if we are searching for weak solutions of semilinear equations, $Lu = f(x,u)$, then the Lax–Milgram theorem no longer applies. Variational methods are often used to circumvent this issue. The key idea is to carefully identify an associated energy functional of the elliptic equation whose critical points are indeed weak solutions of the elliptic problem.

Remark 2.10. *Although variational methods are used to find weak solutions, elliptic regularity theory often ensures that weak solutions are actually strong or classical solutions.*

We begin with a simple example for the sake of illustration. Consider

$$\begin{cases} -\Delta u = f(x) & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{2.29}$$

and consider the functional

$$J(u) = \frac{1}{2}\int_U |Du|^2\,dx - \int_U f(x)u\,dx, \;\; u \in H_0^1(U). \tag{2.30}$$

Remark 2.11. *In general, we will consider the semilinear case when $f = f(x,u)$ case. In the special case where $f(x,u) = |u|^{p-1}u$, then we get the problem*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.31}$$

Equation (2.31) *is often called the Lane-Emden equation. It serves as the prototypical semilinear equation, and it is the model that we will study in great detail throughout these notes. Indeed, the exponent p has important implications in both the quantitative and qualitative properties of solutions and there are three primary cases to consider. In particular, we say the equation is subcritical, critical or super-critical, respectively, if $p < \frac{n+2}{n-2}$, $p = \frac{n+2}{n-2}$ or $p > \frac{n+2}{n-2}$.*

We now show that if u is a minimizer of this functional $J(\cdot)$ in the class of $H_0^1(U)$, then u a weak solution of (2.29). Let v be any function in $H_0^1(U)$ and consider the real-valued function

$$g(t) = J(u+tv), \; t \in \mathbb{R}.$$

Since u is a minimizer of $J(\cdot)$, the function $g(t)$ has a minimum at $t = 0$, and thus we must have

$$0 = g'(0) = \frac{d}{dt} J(u+tv)\Big|_{t=0},$$

where explicitly,

$$J(u+tv) = \frac{1}{2}\int_U |D(u+tv)|^2\,dx - \int_U f(x)(u+tv)\,dx,$$

and

$$\frac{d}{dt}J(u+tv) = \int_U D(u+tv)\cdot Dv\,dx - \int_U f(x)v\,dx.$$

Hence, $g'(0) = 0$ implies

$$\int_U Du \cdot Dv\,dx - \int_U f(x)v\,dx = 0, \;\; \text{for all } \; v \in H_0^1(U),$$

and so u is a weak solution of (2.29).

Remark 2.12. *The first derivative $g'(0)$ is often called the* **first variation** *of $J(\cdot)$. In the next chapter, we develop the regularity theory for the weak solutions of such elliptic problems. In particular, it follows that the weak solution of* (2.29) *obtained by our variational method is a classical solution provided f is regular enough, e.g., it is Hölder continuous.*

Clearly, for u to be a weak solution it need not be a minimum; it can be a maximum or saddle point of the functional, or generally any point that satisfies

$$0 = \frac{d}{dt}J(u+tv)\Big|_{t=0}.$$

This motivates the following definition.

Definition 2.6. *Let $J = J(\cdot)$ be a functional on a Banach space X.*

(a) We say that J is Frechet differentiable at $u \in X$ if there exists a continuous linear map $\mathcal{L} : X \longrightarrow X^*$ satisfying: For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon, u)$ such that

$$|J(u+v) - J(u) - \langle \mathcal{L}(u), v\rangle| \leq \epsilon \|v\|_X \ \text{ whenever } \ \|v\|_X < \delta.$$

The mapping $\mathcal{L}(u)$ is commonly denoted by $J'(u)$.

(b) A critical point of J is a point at which $J'(u) = 0$; that is,

$$\langle J'(u), v\rangle = 0 \ \text{ for all } \ v \in X.$$

We call $J'(u) = 0$, and the PDE associated with this distribution equation, the **Euler-Lagrange** *equation of the functional $J(\cdot)$.*

Remark 2.13. *One can verify that if J is Frechet differentiable at u, then*

$$\langle J'(u), v\rangle = \lim_{t \longrightarrow 0} \frac{J(u+tv) - J(u)}{t} = \frac{d}{dt} J(u+tv)\Big|_{t=0}.$$

More generally, given the **Lagrangian** $L : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \longrightarrow \mathbb{R}$ with $L = L(p, z, x)$ and using the notation

$$\begin{cases} D_p L = (L_{p_1}, \ldots, L_{p_n}), \\ D_z L = L_z, \\ D_x L = (L_{x_1}, \ldots, L_{x_n}), \end{cases}$$

we may consider the functional

$$J(u) = \int_U L(Du(x), u(x), x)\, dx.$$

As before, we may compute the **Euler-Langrange** equation associated with this functional $J(\cdot)$ to be the divergence-form elliptic equation

$$-\sum_{i=1}^{n} (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0 \ \text{ in } \ U.$$

Although we will mostly focus on the special case

$$L(p, z, x) = \frac{1}{2}|p|^2 - z f(x),$$

which corresponds to the functional (2.30), the results we cover extend to more general Lagrangians under some coercivity and convexity assumptions on L (see Chapter 8 in [8]).

2.7.1 Existence of Weak Solutions

We prove the following theorem.

Theorem 2.20. *Suppose that $U \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂U. Then for every $f \in L^{\frac{2n}{n+2}}(U)$ with $n > 2$, the functional*

$$J(u) = \frac{1}{2}\int_U |Du|^2\,dx - \int_U f(x)u\,dx$$

possesses a minimum $u_0 \in H_0^1(U)$, which is a weak solution of the boundary value problem

$$\begin{cases} -\Delta u = f(x) & \text{in } U, \\ \quad\; u = 0 & \text{on } \partial U. \end{cases} \tag{2.32}$$

Proof. Let u_k be a minimizing sequence, i.e.,

$$\inf_{u\in H_0^1(U)} J(u) = \lim_{k\longrightarrow\infty} J(u_k).$$

Our goal is to show there does exist a function $u_0 \in H_0^1(U)$ such that

$$J(u_0) = \lim_{k\longrightarrow\infty} J(u_k) = \inf_{u\in H_0^1(U)} J(u),$$

and as discussed earlier, u_0 is indeed a weak solution of the boundary value problem (2.32). To prove the existence of a minimum of the functional J, there are three main ingredients to verify: the functional J is

1. bounded from below,
2. coercive, and
3. weakly lower semi-continuous on $H_0^1(U)$.

1. We prove that J is bounded from below in $H := H_0^1(U)$ if $f \in L^2(U)$. From Poincaré's inequality, we endow the following equivalent norm on H:

$$\|u\|_H := \left(\int_U |Du|^2\,dx\right)^{1/2}$$

Thus, by Hölder and Poincaré's inequalities, we have

$$J(u) \geq \frac{1}{2}\|u\|_H^2 - C\|u\|_H\|f\|_{L^2(U)} = \frac{1}{2}\left(\|u\|_H - C\|f\|_{L^2(U)}\right)^2 - \frac{C^2}{2}\|f\|^2_{L^2(U)} \geq -\frac{C^2}{2}\|f\|^2_{L^2(U)}.$$

2. Observe that a function bounded below does not guarantee it has a minimum. Take, for instance, $\frac{1}{1+x^2}$ on the real line. For a given minimizing sequence, we must make certain that the sequence does not "leak" to infinity. This motivates our need for a **coercive** condition. That is, if a sequence $\{u_k\}$ tends to infinity, i.e., $\|u_k\|_H \longrightarrow \infty$, then $J(u_k)$ must also become unbounded. In fact, it is clear that $J(u_k) \longrightarrow \infty$ as $\|u_k\|_H \longrightarrow \infty$ for our specific problem. This implies that a minimizing sequence would be retained in a bounded set; that is, any minimizing sequence must be bounded in H.

By the reflexivity of the Hilbert space H and the weak-$*$ compactness of the unit ball, the minimizing sequence has a weakly convergent subsequence, we still denote $\{u_k\}$, in H with limit point $u_0 \in H$. We shall show that u_0 is a minimum point of J.

3. We prove J is weakly lower semi-continuous on H.

Definition 2.7. *We say a functional $J(\cdot)$ is* **weakly lower semi-continuous** *on a Banach space X if for every weakly convergent sequence*

$$u_k \rightharpoonup u_0 \ \text{ in } \ X,$$

we have

$$J(u_0) \leq \liminf_{k \longrightarrow \infty} J(u_k).$$

Clearly, it holds from the definition that $J(u_0) \geq \liminf_{k\longrightarrow\infty} J(u_k)$. Thus, if J is weakly lower semi-continuous, then $J(u_0) = \lim_{k\longrightarrow\infty} J(u_k)$. Hence, u_0 is a minimum of J and this completes the proof of the theorem provided we show J is weakly lower semi-continuous on H. Note that since $f \in L^{\frac{2n}{n+2}}(U)$, Hölder's inequality implies that $u \longrightarrow \int_U f(x)u\,dx$ is a continuous linear functional on H and thus,

$$\int_U f(x)u_k\,dx \longrightarrow \int_U f(x)u_0\,dx \ \text{ as } \ k \longrightarrow \infty. \tag{2.33}$$

From the algebraic inequality $a^2 + b^2 \geq 2ab$, we get $|Du_k|^2 + |Du_0|^2 \geq 2Du_0 \cdot Du_k$ or

$$\int_U |Du_k|^2\,dx + \int_U |Du_0|^2\,dx \geq 2\int_U Du_0 \cdot Du_k\,dx,$$

which after subtracting $2\int_U |Du_0|^2\,dx$ on both sides of this inequality yields

$$\int_U |Du_k|^2\,dx \geq \int_U |Du_0|^2\,dx + 2\int_U Du_0 \cdot (Du_k - Du_0)\,dx.$$

This leads to

$$\liminf_{k\longrightarrow\infty} \int_U |Du_k|^2\,dx \geq \int_U |Du_0|^2\,dx,$$

since

$$\int_U Du_0 \cdot (Du_k - Du_0)\,dx \longrightarrow 0 \ \text{ as } \ k \longrightarrow \infty.$$

Combining this with (2.33) yields the desired result. $\square$

2.7.2 Existence of Minimizers Under Constraints

We extend the previous result to the Lane-Emden equation in the subcritical case.

Theorem 2.21. *Suppose that $U \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂U and let $1 < p < \frac{n+2}{n-2}$. Then there exists a non-trivial weak solution $u \in H_0^1(U)$ of the semi-linear Dirichlet problem*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } U, \\ \quad u = 0 & \text{on } \partial U. \end{cases} \tag{2.34}$$

Remark 2.14. *We must be careful in setting up our variational procedure for this problem. For example, we can naively consider the functional*

$$J(u) = \frac{1}{2}\int_U |Du|^2\,dx - \frac{1}{p+1}\int_U |u|^{p+1}\,dx.$$

It is not to difficult to show that

$$\frac{d}{dt}J(u+tv)\Big|_{t=0} = \int_U Du\cdot Dv - |u|^{p-1}uv\,dx.$$

Therefore, a critical point of the functional J in $H := H_0^1(U)$ is a weak solution of (2.34). However, the functional J is not bounded from below in H. To see this, fix $u \in H$ and consider

$$J(tu) = \frac{t^2}{2}\int_U |Du|^2\,dx - \frac{t^{p+1}}{p+1}\int_U |u|^{p+1}\,dx.$$

Since $p+1 > 2$, we see that $J(tu) \longrightarrow -\infty$ as $t \longrightarrow \infty$. To get around this problem, we choose a different functional with constraints.

Proof. Set

$$I(u) = \frac{1}{2}\int_U |Du|^2\,dx$$

under the constraint

$$M := \{u \in H : G(u) := \int_U |u|^{p+1}\,dx = 1\}.$$

We seek minimizers of I in M. Let $\{u_k\} \subset M$ be a minimizing sequence. It follows that $\int_U |Du_k|^2\,dx$ is bounded so that $\{u_k\}$ is bounded in H. By the weak-$*$ compactness of bounded sets in the reflexive Hilbert space H, u_k converges weakly to some u_0 in H. Thus, the weak lower semi-continuity of the functional I implies that

$$I(u_0) \le \liminf_{k\longrightarrow\infty} I(u_k) =: m. \tag{2.35}$$

Since $p+1 < \frac{2n}{n-2}$, the compact Sobolev embedding theorem implies that $H^1(U)$ is compactly embedded in $L^{p+1}(U)$. Therefore, u_k converges strongly to u_0 in $L^{p+1}(U)$, which implies $u_0 \in M$ since

$$1 = \int_U |u_k|^{p+1}\,dx \longrightarrow \int_U |u_0|^{p+1}\,dx \;\text{ as }\; k \longrightarrow \infty.$$

Thus, $I(u_0) \geq m$. Combining this with (2.35) yields $I(u_0) = m$. This proves the existence of a minimizer u_0 of I in M. It remains to show that u_0, multiplied by a suitable constant if necessary, is a non-trivial weak solution of (2.34). This entails identifying the corresponding Euler–Lagrange equation for this minimizer under the constraint, which is provided by the following theorem whose proof is given on page 60 in [5].

Theorem 2.22 (Lagrange Multiplier). *Let u be a minimizer of I in M, i.e.,*

$$I(u) = \min_{v \in M} I(v).$$

Then there exists a real number λ such that

$$I'(u) = \lambda G'(u)$$

or

$$\langle I'(u), v\rangle = \lambda \langle G'(u), v\rangle \;\text{ for all }\; v \in H.$$

We are now ready to show the minimizer u_0 is a weak solution of (2.34) after a suitable dilation. The minimizer u_0 of I under the constraint $G(u) = 1$ satisfies the Euler–Lagrange equation

$$\langle I'(u_0), v\rangle = \lambda \langle G'(u_0), v\rangle \;\text{ for all }\; v \in H;$$

that is,

$$\int_U Du_0 \cdot Dv\,dx = \lambda \int_U |u_0|^{p-1} u_0 v\,dx \;\text{ for all }\; v \in H.$$

From this, we can choose $v = u_0$ so that

$$\lambda = \frac{\displaystyle\int_U |Du_0|^2\,dx}{\displaystyle\int_U |u_0|^{p+1}\,dx},$$

and thus $\lambda > 0$. Then we can set $\tilde{u} = a u_0$ where $\lambda / a^{p-1} = 1$ since $p > 1$. Hence

$$\int_U D\tilde{u} \cdot Dv\,dx = \int_U |\tilde{u}|^{p-1} \tilde{u} v\,dx,$$

so $\tilde{u} \in H$ is a weak solution of (2.34). $\square$

2.8 Calculus of Variations II: Critical Points and the Mountain Pass Theorem

In the previous examples, we obtained minimizers to a certain functional, which are weak solutions to its corresponding PDE. More generally, we also showed that the critical points of the functional are also weak solutions. In this section, we use the celebrated Mountain Pass theorem of Ambrosetti and Rabinowitz (see [1]) to find these critical points, which are often times saddle points rather than minimizers or maximizers. In order to state and prove the Mountain Pass theorem, we first need to introduce some definitions and an important deformation theorem.

2.8.1 The Deformation and Mountain Pass Theorems

Hereafter, H denotes a Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$ and $I : H \longrightarrow \mathbb{R}$ is a nonlinear functional on H.

Definition 2.8. *We say I is differentiable at $u \in H$ if there exists $v \in H$ such that*

$$I[w] = I[u] + (v, w-u) + o(\|w-v\|) \text{ for } w \in H. \tag{2.36}$$

The element v, if it exists, is unique and we write $I'[u] = v$.

Definition 2.9. *We say I belongs to $C^1(H;\mathbb{R})$ if $I'[u]$ exists for each $u \in H$ and the mapping $I' : H \longrightarrow H$ is continuous.*

Remark 2.15. *(a) The results we develop in this section holds if $I \in C^1(H : \mathbb{R})$, but for simplicity, we shall additionally assume that $I' : H \longrightarrow H$ is Lipschitz continuous on bounded subsets of H. Moreover, we denote by $\mathcal{C}$ the collection of such I satisfying these conditions.*

(b) If $c \in \mathbb{R}$, we set

$$A_c := \{u \in H \mid I[u] \le c\} \text{ and } K_c := \{u \in H \mid I[u] = c,\ I'[u] = 0\}.$$

Definition 2.10. *We say $u \in H$ is a critical point if $I'[u] = 0$. The real number c is a critical value if $K_c \neq \emptyset$.*

In general, H is taken to be an infinite dimensional Hilbert space, thus we need to impose some sort of compactness condition.

Definition 2.11 (Palais-Smale)**.** *A functional $I \in C^1(H;\mathbb{R})$ satisfies the Palais-Smale compactness condition if each sequence $\{u_k\}_{k=1}^{\infty} \subset H$ such that*

(a) $\{I[u_k]\}_{k=1}^{\infty}$ is bounded,

(b) $I'[u_k] \longrightarrow 0$ *in* H*,*

is precompact in H*.*

The following theorem states that if c is not a critical value, we can deform the set $A_{c+\epsilon}$ into $A_{c-\epsilon}$ for some $\epsilon > 0$. The principle idea lies around solving an ODE in H.

Theorem 2.23 (Deformation)**.** *Assume* $I \in \mathcal{C}$ *satisfies the Palais-Smale condition and suppose that* $K_c = \emptyset$*. Then for each sufficiently small* $\epsilon > 0$*, there exists a constant* $\delta \in (0, \epsilon)$ *and a deformation function*

$$\eta \in C([0,1] \times H; H)$$

such that the mappings

$$\eta_t(u) = \eta(t, u) \;\; \text{for} \;\; t \in [0,1],\, u \in H$$

satisfy

(i) $\eta_0(u) = u$ *for* $u \in H$*,*

(ii) $\eta_1(u) = u$ *for* $u \notin I^{-1}[c-\epsilon,\, c+\epsilon])$*,*

(iii) $I[\eta_t(u)] \leq I[u]$ *for* $t \in [0,1]$*,* $u \in H$*,*

(iv) $\eta_1(A_{c+\delta}) \subset A_{c-\delta}$*.*

Proof. **Step 1:** We claim that there exist constants $\sigma, \epsilon \in (0,1)$ such that

$$\|I'[u]\| \geq \sigma \;\; \text{for each} \;\; u \in A_{c+\epsilon} - A_{c-\epsilon}. \tag{2.37}$$

To see this, we proceed by contradiction. Assume (2.37) were false for all constant $\sigma, \epsilon > 0$. Then there would exist sequences $\sigma_k \to 0$ and $\epsilon_k \to 0$ and elements

$$u_k \in A_{c+\epsilon_k} - A_{c-\epsilon_k} \;\; \text{with} \;\; \|I'[u_k]\| \leq \sigma_k. \tag{2.38}$$

According to the Palais-Smale condition, there is a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and an element $u \in H$ such that $u_k \to u$ in H. Since $I \in C^1(H;\mathbb{R})$, (2.38) implies that $I[u] = c$ and $I'[u] = 0$. Hence, $K_c \neq \emptyset$ and we arrive at a contradiction.

Step 2: Now fix δ such that

$$\delta \in (0, \epsilon) \;\; \text{and} \;\; \delta \in (0, \sigma^2/2). \tag{2.39}$$

Denote

$$\begin{aligned} A &:= \{u \in H \,|\, I[u] \leq c - \epsilon \;\; \text{or} \;\; I[u] \geq c + \epsilon\}, \\ B &:= \{u \in H \,|\, c - \delta \leq I[u] \leq c + \delta\}. \end{aligned}$$

Since I' is bounded on bounded sets, we verify that the mapping $u \mapsto dist(u,A)+dist(u,B)$ is bounded from below by a positive constant on each bounded subset of H. Therefore, the function,

$$g(u) = \frac{dist(u,A)}{dist(u,A)+dist(u,B)}, \;\; (u \in H),$$

is Lipschitz continuous on bounded sets and satisfies

$$0 \le g \le 1, \, g = 0 \;\text{ on }\; A, \, g = 1 \;\text{ on }\; B.$$

Now set

$$h(t) = \begin{cases} 1, & \text{if } t \in [0,1], \\ 1/t, & \text{if } t \ge 1, \end{cases} \tag{2.40}$$

and define the bounded operator $V : H \to H$ by

$$V(u) = -g(u)h(\|I'[u]\|)I'[u] \, (u \in H). \tag{2.41}$$

Consider, for each $u \in H$, the abstract ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}\eta(t) = V(\eta(t)) \quad t > 0, \\ \quad \eta(0) = u. \end{cases} \tag{2.42}$$

Indeed, there exists a unique global solution $\eta = \eta(t,u) = \eta_t(u)$ for $t \ge 0$, since V is bounded and Lipschitz continuous on bounded sets. Moreover, if we restrict our attention to the smaller interval $t \in [0,1]$, it is easy to see that $\eta \in C([0,1] \times H; H)$ and satisfies assertions (i) and (ii).

Step 3: It remains to verify assertions (iii) - (iv).
There holds

$$\frac{d}{dt}I[\eta_t(u)] = I'[\eta_t(u)] \cdot \frac{d}{dt}\eta_t(u) = I'[\eta_t(u)] \cdot V(\eta_t(u)) = -g(\eta_t(u))h(\|I'[\eta_t(u)]\|)\|I'[\eta_t(u)]\|^2. \tag{2.43}$$

In particular,

$$\frac{d}{dt}I[\eta_t(u)] \le 0 \;\text{ for }\; u \in H, \, t \in [0,1],$$

and this verifies assertion (iii).

Now fix any point $u \in A_{c+\delta}$. We claim that $\eta_1(u) \in A_{c-\delta}$, i.e., assertion (iv) holds. To see this, if $\eta_t(u) \notin B$ for some $t \in [0,1]$, we are done. So, instead, assume that $\eta_t(u) \in B$ for all $t \in [0,1]$. Then $g(\eta_t(u)) = 1$ for all $t \in [0,1]$. Hence, identity (2.43) implies that

$$\frac{d}{dt}I[\eta_t(u)] = -h(\|I'[\eta_t(u)]\|)\|I'[\eta_t(u)]\|^2. \tag{2.44}$$

If $\|I'[\eta_t(u)]\| \ge 1$, then (2.37) and (2.40) imply that

$$\frac{d}{dt}I[\eta_t(u)] = -\|I'[\eta_t(u)]\|^2 \le -\sigma^2.$$

Likewise, if $\|I'[\eta_t(u)]\| \leq 1$, then (2.37) and (2.40) also imply that

$$\frac{d}{dt} I[\eta_t(u)] \leq -\sigma^2.$$

These two inequalities, when combined with (2.39) and (2.44), imply

$$I[\eta_1(u)] \leq I[u] - \sigma^2 \leq c + \delta - \sigma^2 \leq c - \delta.$$

This verifies the claim that $\eta_1(u) \in A_{c-\delta}$ and this completes the proof. $\square$

With the help of the Deformation Theorem, we shall now prove the celebrated Mountain Pass Theorem, which guarantees the existence of a critical point.

Theorem 2.24 (Mountain Pass)**.** *Assume $I \in \mathcal{C}$ satisfies the Palais-Smale condition. Suppose, in addition, that*

(i) $I[0] = 0$,

(ii) there exist constants $a, r > 0$ such that

$$I[u] \geq a \quad \text{if} \quad \|u\| = r,$$

(iii) there exists an element $v \in H$ with

$$\|v\| > r,\ I[v] \leq 0.$$

Then

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I[g(t)],$$

where

$$\Gamma := \{g \in C([0,1]; H) \,|\, g(0) = 0,\ g(1) = v\},$$

is a critical value of I.

Proof. Indeed, it is clear that $c \geq a$. Now assume that c is not a critical value of I so that $K_c = \emptyset$. Choose a suitably small $\epsilon \in (0, a/2)$. According to the deformation theorem, there exists a constant $\delta \in (0, \epsilon)$ and a homeomorphism $\eta : H \to H$ with

$$\eta(A_{c+\delta}) \subset A_{c-\delta}$$

and

$$\eta(u) = u \quad \text{if} \quad u \notin I^{-1}[c - \epsilon, c + \epsilon]. \tag{2.45}$$

Now select $g \in \Gamma$ such that

$$\max_{0 \leq t \leq 1} I[g(t)] \leq c + \delta. \tag{2.46}$$

Then the composition

$$\hat{g} = \eta \circ g$$

is also in Γ, since $\eta(g(0)) = \eta(0) = 0$ and $\eta(g(1)) = \eta(v) = v$ as indicated in (2.45). But then (2.46) implies that

$$\max_{0 \le t \le 1} I[\hat{g}(t)] \le c - \delta,$$

and so

$$c = \inf_{g \in \Gamma} \max_{0 \le t \le 1} I[g(t)] \le c - \delta,$$

which is a contradiction. □

2.8.2 Application of the Mountain Pass Theorem

We will prove the existence of at least one non-trivial weak solution to a general semilinear boundary value problem in which the Lane-Emden equation is a special case. Namely, consider the boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } U, \\ \quad u = 0 & \text{on } \partial U. \end{cases} \tag{2.47}$$

We assume f is smooth, and for some $1 < p < \frac{n+2}{n-2}$, there holds for some positive constant C,

$$|f(z)| \le C(1 + |z|^p),\ |f'(z)| \le C(1 + |z|^{p-1}) \text{ for } z \in \mathbb{R}. \tag{2.48}$$

If we denote

$$F(z) = \int_0^z f(s)\, ds \text{ and } z \in \mathbb{R},$$

we also assume that

$$0 \le F(z) \le \gamma f(z) z \text{ for some constant } \gamma < 1/2, \tag{2.49}$$

and for constants $0 < a \le A$,

$$a|z|^{p+1} \le |F(z)| \le A|z|^{p+1} \text{ for } z \in \mathbb{R}. \tag{2.50}$$

Remark 2.16. *(a) Indeed,* (2.50) *implies that* $f(0) = 0$ *and so* $u \equiv 0$ *is a trivial solution of* (2.47).

(b) It is easy to check that $f(u) = |u|^{p-1}u$ *satisfies the above conditions.*

Theorem 2.25. *The boundary value problem* (2.47) *has at least one non-trivial weak solution.*

The basic idea of the proof is to consider the functional

$$I[u] := \int_U \frac{1}{2}|Du|^2 - F(u)\,dx \ \text{ for } u \in H, \tag{2.51}$$

where $H = H_0^1(U)$ with the induced norm coming from the inner product $(u,v) = \int_U Du \cdot Dv\,dx$, then show that the Mountain Pass Theorem applies. Therefore, the existence of a non-trivial critical point of I implies the existence of a non-trivial weak solution of the boundary value problem. To best illustrate the main ingredients of the proof, we introduce the following lemmas.

Lemma 2.5. *There hold $I[0] = 0$ and I belongs to the class $\mathcal{C}$.*

Proof. It is obvious that $I[0] = 0$. It remains to show that $I \in \mathcal{C}$. Consider the splitting

$$I[u] = \frac{1}{2}\|u\|^2 - \int_U F(u)\,dx := I_1[u] + I_2[u].$$

Indeed, for $u, w \in H$,

$$I_1[w] = \frac{1}{2}\|w\|^2 = \frac{1}{2}\|u+w-u\|^2 = \frac{1}{2}\|u\|^2 + (u, w-u) + \frac{1}{2}\|w-u\|^2 = I_1[u] + (u, w-u) + o(\|w-u\|).$$

Therefore, I_1 is differentiable at u with $I_1'[u] = u$, and thus $I_1 \in \mathcal{C}$. Now we show $I_2 \in \mathcal{C}$. First we make some preliminary observations. Recall that the Lax-Milgram theorem states that for each element $v^* \in H^{-1}(U)$, the boundary value problem,

$$\begin{cases} -\Delta v = v^* & \text{in } U, \\ \quad\; v = 0 & \text{on } \partial U. \end{cases} \tag{2.52}$$

has a unique solution $v \in H_0^1(U)$. Write $v = Kv^*$ so that

$$K : H^{-1}(U) \to H_0^1(U) \tag{2.53}$$

is an isometry. In particular, recall that if $w \in L^{\frac{2n}{n+2}}(U)$, then the linear functional w^* defined by

$$(w^*,\, u) := \int_U wu\,dx \ \text{ for } \ u \in H_0^1(U)$$

belongs to $H^{-1}(U)$. Here we shall abuse conventional notation and say that w belongs to $H^{-1}(U)$. In addition, the subcritical condition implies that $p(\frac{2n}{n+2}) < \frac{2n}{n-2}$ and so $f(u)$ belongs to $L^{\frac{2n}{n+2}}(U) \subset H^{-1}(U)$ provided that $u \in H_0^1(U)$. The crucial step here is that

$$I_2'[u] = K[f(u)]. \tag{2.54}$$

To see this, notice that

$$F(a+b) = F(a) + f(a)b + \int_0^1 (1-s)f'(a+sb)\,dsb^2$$

and thus for each $w \in H_0^1(U)$,

$$
\begin{aligned}
I_2[w] &= \int_U F(w)\,dx = \int_U F(u+w-u)\,dx = \int_U F(u) + f(u)(w-u)\,dx + R \\
&= I_2(u) + \int_U DK[f(u)] \cdot D(w-u)\,dx + R,
\end{aligned}
$$

where the remainder term R, according to (2.48), satisfies

$$
\begin{aligned}
|R| &\leq C \int_U (1 + |u|^{p-1} + |w-u|^{p-1})|w-u|^2\,dx \\
&\leq C_1\Big(\int_U |w-u|^2 + |w-u|^{p+1}\,dx\Big) + C_2\Big(\int_U |u|^{p+1}\,dx\Big)^{\frac{p-1}{p+1}}\Big(\int_U |w-u|^{p+1}\,dx\Big)^{\frac{2}{p+1}}.
\end{aligned}
$$

Hence, since $p+1 < \frac{2n}{n-2}$, Sobolev embedding implies that $R = o(\|w-u\|)$. Therefore,

$$
I_2[w] = I_2[w] + (K[f(u)],\, w-u) + o(\|w-u\|).
$$

Lastly, if $u, v \in B_L(0) \subset H_0^1(U)$, then

$$
\|I_2'[u] - I_2'[v]\| = \|K[f(u)] - K[f(v)]\|_{H_0^1(U)} = \|f(u) - f(v)\|_{H^{-1}(U)} \leq \|f(u) - f(v)\|_{L^{\frac{2n}{n+2}}}.
$$

Furthermore, (2.48) and Hölder's inequality imply

$$
\begin{aligned}
\|f(u) - f(v)\|_{L^{\frac{2n}{n+2}}(U)} &\leq C\Big(\int_U ((1 + |u|^{p-1} + |v|^{p-1})|u-v|)^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{2n}} \\
&\leq C\Big(\int_U ((1 + |u|^{p-1} + |v|^{p-1})|u-v|)^{\frac{2n}{n+2}\frac{n+2}{4}}\,dx\Big)^{\frac{2}{n}} \|u-v\|_{L^{\frac{2n}{n-2}}(U)} \\
&\leq C(L)\|u-v\|_{L^{\frac{2n}{n-2}}(U)} \\
&\leq C(L)\|u-v\|.
\end{aligned}
$$

This shows that $I_2' : H_0^1(U) \to H_0^1(U)$ is Lipschitz continuous on bounded sets and thus, $I_2 \in \mathcal{C}$. This completes the proof of the lemma. $\square$

Lemma 2.6. *The functional $I \in \mathcal{C}$ satisfies the Palais-Smale condition.*

Proof. Suppose the sequence $\{u_k\}_{k=1}^\infty$ in $H_0^1(U)$ satisfies

$$
(i)\ \{I[u_k]\}_{k=1}^\infty \text{ is bounded, and } (ii)\ I'[u_k] \to 0 \text{ in } H_0^1(U). \tag{2.55}
$$

Obviously, we have that

$$
u_k - K(f(u_k)) \to 0 \text{ in } H_0^1(U). \tag{2.56}
$$

Thus, for each $\epsilon > 0$, we have

$$|(I'[u_k],\, v)| = \Big| \int_U Du_k \cdot Dv - f(u_k)v\, dx \Big| \leq \epsilon \|v\| \ \text{ for } \ v \in H_0^1(U)$$

for sufficiently large k. Namely, if we take $v = u_k$ and set $\epsilon = 1$, then we get

$$\Big| \int_U |Du_k|^2 - f(u_k)u_k\, dx \Big| \leq \|u_k\|$$

for sufficiently large k. From (2.55), we have that

$$\Big(\frac{1}{2}\|u_k\|^2 - \int_U F(u_k)\, dx\Big) \leq C < \infty$$

for all k. Hence, we deduce from above and (2.49) that

$$\|u_k\|^2 \leq C + 2\int_U F(u_k)\, dx \leq C + 2\gamma(\|u_k\|^2 + \|u_k\|).$$

As $2\gamma < 1$, we can absorb the last two terms on the right-hand side by the left-hand side to get that $\{u_k\}_{k=1}^{\infty}$ is bounded in $H_0^1(U)$. We can then extract a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$, that converges weakly to $u \in H_0^1(U)$. Hence, $u_{k_j} \longrightarrow u$ in $L^{p+1}(U)$ since $p+1 < \frac{2n}{n-2}$ by the compact Sobolev embedding. But then $f(u_{k_j}) \longrightarrow f(u)$ in $H^{-1}(U)$ and so $K[f(u_{k_j})] \longrightarrow K[f(u)]$ in $H_0^1(U)$. Consequently, from (2.56), we arrive at the desired conclusion that

$$u_{k_j} \to u \ \text{ in } \ H_0^1(U). \tag{2.57}$$

□

Lemma 2.7. *There hold the following statements.*

(a) There exist constants $r, a > 0$ such that

$$I[u] \geq a \ \text{ if } \ \|u\| = r.$$

(b) There exists an element $v \in H_0^1(U)$ with

$$\|v\| > r \ \text{ and } \ I[v] \leq 0.$$

Proof. (i) Suppose that $u \in H_0^1(U)$ with $\|u\| = r$ for some $r > 0$ to be determined below. Then

$$I[u] = I_1[u] - I_2[u] = \frac{r^2}{2} - I_2[u].$$

By (2.50) and Sobolev embedding, as $p+1 < \frac{2n}{n-2}$, we obtain that

$$|I_2[u]| \leq C\int_U |u|^{p+1}\, dx \leq C\Big(\int_U |u|^{\frac{2n}{n-2}}\, dx\Big)^{\frac{(p+1)(n-2)}{2n}} \leq C\|u\|^{p+1} \leq Cr^{p+1}.$$

Hence,

$$I[u] \geq \frac{r^2}{2} - Cr^{p+1} \geq \frac{r^2}{4} = a > 0,$$

provided that $r > 0$ is chosen small enough, since $p + 1 > 2$.

(ii) Fix some non-trivial element $u \in H_0^1(U)$ and write $v = tu$ for $t > 0$ to be determined below. Then, using (2.50), we get

$$I[v] = I_1[tu] - I_2[tu] = t^2 I_1[u] - \int_U F(tu)\,dx \leq t^2 I_1[u] - at^{p+1} \int_U |u|^{p+1}\,dx < 0$$

for $t > 0$ large enough. $\square$

Proof of Theorem 2.25. Indeed, Lemmas 2.5–2.7 verify all the hypotheses in the Mountain Pass theorem. Hence, the Mountain Pass theorem implies there exists a non-trivial function $u \in H_0^1(U)$ with

$$I'[u] = u - K[f(u)] = 0.$$

In particular, for each $v \in H_0^1(U)$, there holds

$$\int_U Du \cdot Dv\,dx = \int_U f(u)v\,dx,$$

and so u is a non-trivial weak solution of the boundary value problem (2.47). $\square$

2.9 Calculus of Variations III: Concentration Compactness

In our variational approach for establishing the existence of solutions to semilinear equations, we exploited the compact Sobolev embedding due to the subcritical exponent p. In the critical setting, however, this compactness property fails. Fortunately, we can apply the principle of concentration compactness to recover the compactness of the minimizing sequence in the strong topology of $H_0^1(U)$. In Chapter 6, we look at this precise problem of concentration phenomena and how it relates to the breakdown of the compactness of critical Sobolev embeddings. More precisely, there we examine finding extremal functions to a constrained energy functional for a critical Sobolev inequality. Then we use the concentration compactness principle to recover strong convergence of a minimizing sequence to obtain a minimizer for the functional.

For now, we illustrate how to apply the concentration compactness principle to establish an existence result for a model elliptic problem. Namely, we consider the stationary Schrödinger equation

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1}u \text{ in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{2.58}$$

where $n \geq 3$, $\lambda < 0$ and $p > 1$.

We first begin with some background and motivation. The well-known nonlinear Schrödinger (NLS) equation is given by

$$\begin{cases} i\partial_t v + \Delta v = \pm |v|^{p-1}v & (x,t) \text{ in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = \varphi(x) & \text{in } H_0^1(\mathbb{R}^n), \end{cases} \tag{2.59}$$

where solutions are understood in the usual weak or distributional sense. We say the nonlinearity in equation (2.59) is focusing or defocusing, respectively, if the right-hand side is $-|v|^{p-1}v$ or $+|v|^{p-1}v$, but we shall only concern ourselves with the focusing case. In either case, however, a key feature of the NLS equation is that mass and energy are conserved quantities, i.e., $M(v(t)) = M(v(0))$ and $E(v(t)) = M(v(0))$ where

$$M(v(t)) = \int_{\mathbb{R}^n} |v(x,t)|^2 \, dx$$

and

$$E(v(t)) = \int_{\mathbb{R}^n} |Dv(x,t)|^2 \, dx \pm \frac{1}{p+1} \int_{\mathbb{R}^n} |v(x,t)|^{p+1} \, dx.$$

In the focusing case, we may search for solitary wave solutions of the form $v(x,t) = u(x)e^{-i\lambda t}$ where u is some function in $H^1(\mathbb{R}^n)$ and $\lambda < 0$. Then, it is simple to see that u satisfies

$$-\Delta u = \lambda u + |u|^{p-1}u \;\text{ in }\; \mathbb{R}^n. \tag{2.60}$$

Indeed, there does exist solutions to equation (2.60) whenever $1 < p < (n+2)/(n-2)$, and this can be established through various ODE or variational approaches. For the sake of illustration and to keep our presentation simple, we employ the concentration compactness principle of P. Lions to solve a closely related variational problem. Namely, for $n \geq 3$ and $1 < p < 1 + 4/n$, we look for minimizers of the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |Du|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx$$

under the constraint $\|u\|_2^2 = \lambda$ for a fixed $\lambda > 0$. More precisely, we consider

$$I_\lambda = \inf\{E(u) \,|\, u \in H^1(\mathbb{R}^n),\, \|u\|_2^2 = \lambda\}. \tag{2.61}$$

We establish

Theorem 2.26. *Let $n \geq 3$ and let $p \in (1, 1+4/n)$ and $\lambda > 0$ be arbitrary. Then $I_\lambda > -\infty$ and for any minimizing sequence $\{u_k\}_{k=1}^\infty \subset H^1(\mathbb{R}^n)$ of (2.61), there exists a sequence of points $\{y_k\}_{k=1}^\infty \subset \mathbb{R}^n$ such that the translated sequence $\{u_k(\cdot + y_k)\}_{k=1}^\infty$ is relatively compact in $H^1(\mathbb{R}^n)$ and whose limit is a minimizer of $E(\cdot)$.*

Remark 2.17. *(a) If $1 < p < 1 + 4/n$ and for any $\lambda > 0$, we have that $I_\lambda < 0$ and is finite.*

(b) *Unfortunately, if $p > 1+4/n$, then $I_\lambda = -\infty$ for any $\lambda > 0$, i.e., the energy functional is no longer bounded from below (and this illustrates the restriction on p). For $1+4/n \le p < (n+2)/(n-2)$, we can circumvent this issue by minimizing a slightly different functional (see Theorem 2.31). For another similar problem that minimizes the Dirichlet integral over an appropriately chosen admissible set, we refer the reader to Section 6.3 in Chapter 6.*

(c) *These minimizers for $E(\cdot)$ are indeed weak solutions to equation* (2.60) *but for a completely different parameter λ. In particular, the parameter λ in the problem for I_λ ($\lambda > 0$) and equation* (2.60) *($\lambda < 0$) are not the same and are opposite in sign.*

We shall make use of the following concentration compactness principle which we state without proof [17, 18]. Essentially, this proposition asserts that there are three possibilities when given a bounded sequence in $H^1(\mathbb{R}^n)$. The usual strategy for our variational problem is to verify that the other two "bad" scenarios cannot happen and that only strong precompactness of the sequence must hold.

Proposition 2.1. *Let $\lambda > 0$ and suppose $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $H^1(\mathbb{R}^n)$ such that $\|u_k\|_2^2 = \lambda$ ($k = 1, 2, 3, \ldots$). Then there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ satisfying one of the following three properties.*

(I) *(Compactness) There exists $\{y_j\}_{j=1}^\infty \subset \mathbb{R}^n$ such that for any $\epsilon > 0$, there exists $R > 0$ for which*

$$\int_{y_j+B_R(0)} u_{k_j}^2(x+y_j)\,dx \ge \lambda - \epsilon \text{ for } j = 1, 2, 3, \ldots.$$

(II) *(Vanishing) For all $R > 0$,*

$$\lim_{j\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R(0)} u_{k_j}^2(x)\,dx = 0.$$

(III) *(Dichotomy) There exist $\alpha \in (0, \lambda)$ and bounded sequences $\{u_j^1\}_{j=1}^\infty$ and $\{u_j^2\}_{j=1}^\infty$ in $H^1(\mathbb{R}^n)$ such that*

(a) $\lim_{j\to\infty} \|u_{k_j} - (u_j^1 + u_j^2)\|_q \longrightarrow 0$ *for* $2 \le q < \dfrac{2n}{n-2}$;

(b) $\alpha = \lim_{j\to\infty} \|u_j^1\|_{L^2(\mathbb{R}^n)}^2$ *and* $\lambda - \alpha = \lim_{j\to\infty} \|u_j^2\|_{L^2(\mathbb{R}^n)}^2$;

(c) $\liminf_{j\to\infty} \int_{\mathbb{R}^n} \left\{ |Du_{k_j}|^2 - |Du_j^1|^2 - |Du_j^2|^2 \right\} dx \ge 0.$

Remark 2.18. *Roughly speaking, only three situations can occur for such a bounded sequence of functions. Either (I) the sequence of functions concentrate near the points $\{y_j\}$, (II) such concentration does not occur at any of the points $\{y_j\}$, or (III) some fraction $\lambda \in (0, 1)$ concentrates near some points $\{y_j\}$ while the remaining part spreads away from these points.*

We shall also require the following intermediate result.

Lemma 2.8. *There holds $I_\lambda < I_\alpha + I_{\lambda-\alpha}$ for any $\lambda > 0$ and $\alpha \in (0, \lambda)$.*

Proof. Let $\alpha \in [\lambda/2, \lambda)$ and $\theta \in (1, \lambda/\alpha]$. Then

$$\begin{aligned} I_{\theta\alpha} &= \inf_{u \in H^1(\mathbb{R}^n), \|u\|^2_{L^2(\mathbb{R}^n)} = \theta\alpha} E(u) = \inf_{u \in H^1(\mathbb{R}^n), \|u\|^2_{L^2(\mathbb{R}^n)} = \alpha} E(\theta^{1/2} u) \\ &= \theta \inf_{u \in H^1(\mathbb{R}^n), \|u\|^2_{L^2(\mathbb{R}^n)} = \alpha} \left\{ E(u) - \frac{\theta^{(p-1)/2}}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}\, dx \right\} \\ &< \theta I_\alpha, \end{aligned}$$

where we used the fact that $I_\alpha < 0$ as indicated in Remark 2.17. Hence,

$$I_\lambda < \frac{\lambda}{\alpha} I_\alpha = I_\alpha + \frac{\lambda - \alpha}{\alpha} I_\alpha \leq I_\alpha + I_{\lambda-\alpha}$$

□

Proof of Theorem 2.26. We divide the proof into three main steps.

Step 1: Let $\{u_k\}_{k=1}^\infty$ be a minimizing sequence for the energy functional $E(\cdot)$. The boundedness of the minimizing sequence follows immediately since the sequences $\{E(u_k)\}_{k=1}^\infty$ and $\{\|Du_k\|_{L^2(\mathbb{R}^n)}\}_{k=1}^\infty$ are bounded. From the concentration compactness principle of Proposition 2.1, there are three possibilities that may occur. The goal is to show that (II) vanishing and (III) dichotomy do not happen and that (I) compactness occurs. Once this is verified, the result follows accordingly. Namely, as done in the preceding sections, we may exploit the structure of the energy functional $E(u)$ to show the strong precompactness of the minimizing sequence, i.e., the translated subsequence given in (I) converges to some $u \in H^1(\mathbb{R}^n)$ with $\|u\|^2_{L^2(\mathbb{R}^n)} \leq \lambda$. As usual, the next step is to show that the limit point u is admissible, i.e., $\|u\|^2_{L^2(\mathbb{R}^n} = \lambda$, but this is immediately deduced from case (I) of Proposition 2.1 and we are done. Thus, it only remains to show that (II) and (III) cannot happen.

Step 2: (III) dichotomy does not occur.

Assume the contrary. Let $\alpha_j > 0$ and $\beta_j > 0$ be such that $\|\alpha_j u_j^1\|^2_{L^2(\mathbb{R}^n)} = \alpha$ and $\|\beta_j u_j^2\|^2_{L^2(\mathbb{R}^n)} = \lambda - \alpha$. Then $\alpha_j, \beta_j \longrightarrow 1$ as $j \longrightarrow \infty$ and we have

$$E(u_{k_j}) \geq E(u_j^1) + E(u_j^2) + \gamma_j = E(\alpha_j u_j^1) + E(\beta_j u_j^2) + \gamma_j'$$

where $\gamma_j, \gamma_j' \longrightarrow 0$ as $j \longrightarrow \infty$. Hence,

$$I_\lambda = \lim_{j \to \infty} E(u_{k_j}) \geq \lim_{j \to \infty} [E(\alpha_j u_j^1) + E(\beta_j u_j^2)] \geq I_\alpha + I_{\lambda-\alpha},$$

but this contradicts with Lemma 2.8.

Step 3: (II) vanishing does not occur.

Assume otherwise. It suffices to show that if (II) holds, then $\|u_{k_j}\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} \longrightarrow 0$ as $j \longrightarrow \infty$ because then $\liminf_{j\to\infty} E(u_{k_j}) \geq 0$ and we get a contradiction with the fact that $I_\lambda < 0$. Choose an arbitrary $R > 0$. For any $y \in \mathbb{R}^n$, the Sobolev inequality yields

$$\|u\|^{p+1}_{L^{p+1}(B_R(0))} \leq C(R)\Big(\|u\|^{p+1}_{L^2(y+B_R(0))} + \|u\|^{p+1+n-\frac{n}{2}(p+1)}_{L^2(y+B_R(0))}\|Du\|^{\frac{n}{2}(p+1)-n}_{L^2(y+B_R(0))}\Big).$$

Choose a sequence $\{z_r\}_{r=1}^\infty \subset \mathbb{R}^n$ such that

$$\mathbb{R}^n \subset \bigcup_{r=1}^\infty \{z_r + B_R(0)\}$$

and each point $x \in \mathbb{R}^n$ is contained in at most ℓ balls where ℓ is a fixed positive integer. Then, noting that $\epsilon_j := \sup_r \|u_{k_j}\|^{p-1}_{L^2(z_r+B_R(0))} \longrightarrow 0$ as $j \longrightarrow \infty$ and applying the preceding Sobolev inequality, we get

$$\begin{aligned}
\|u_{k_j}\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} \leq &\sum_{r=1}^\infty \|u_{k_j}\|^{p+1}_{L^{p+1}(z_r+B_R(0))} \\
&\leq C(R)\epsilon_j \sum_{r=1}^\infty \Big\{\|u_{k_j}\|^2_{L^2(z_r+B_R(0))} + \|u_{k_j}\|^{2+n-\frac{n}{2}(p+1)}_{L^2(z_r+B_R(0))}\|Du_{k_j}\|^{\frac{n}{2}(p+1)-n}_{L^2(z_r+B_R(0))}\Big\} \\
&\leq C\epsilon_j \sum_{r=1}^\infty \int_{z_r+B_R(0)} [u^2_{k_j} + |Du_{k_j}|^2]\,dx \leq C\ell\epsilon_j\|u_{k_j}\|^2_{L^1(\mathbb{R}^n)} \longrightarrow 0
\end{aligned}$$

as $j \longrightarrow \infty$, where we used Jensen's inequality in the last line. This proves the claim.

Hence, u is a minimizer of $E(\cdot)$, i.e., $E(u) = I_\lambda$ as defined in problem (2.61). This completes the proof. □

2.10 Sharp Existence Results for Semilinear Equations

We examine, in more detail, existence results for the semilinear problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{2.62}$$

In this section, we discuss how the existence results obtained by the calculus of variations are indeed optimal. We will also study how the geometry and topology of the domain influences the existence and non-existence of solutions. For instance, the existence result of Theorem 2.21 is sharp in that the equation admits no classical non-trivial solution in the super-critical case. Thus, the only solution is indeed the trivial one.

Theorem 2.27. *Let $p > (n+2)/(n-2)$ and $U \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary. Further suppose U is a star-shaped domain with respect to the origin. If $u \in C^2(U) \cap C^1(\bar{U})$ is a solution of (2.62), then it must necessarily be the trivial solution $u \equiv 0$.*

For completeness sake, we include the sketch of the proof, which centers on the following Rellich-Pohozaev identity.

Proposition 2.2. *Let $U \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary and star-shaped with respect to the origin. If $u \in C^2(U) \cap C^1(\bar{U})$ is a solution of* (2.62) *with $p > 1$, then*

$$\frac{n-2}{2} \int_U |Du|^2 \, dx + \frac{1}{2} \int_{\partial U} |Du|^2 (x \cdot \nu) \, dS = \frac{n}{1+p} \int_U |u|^{p+1} \, dx. \tag{2.63}$$

Proof. Multiplying the PDE by $x \cdot Du$ then integrating over U gives us

$$\int_U (x \cdot Du)(-\Delta) u \, dx = \int_U (x \cdot Du) |u|^{p-1} u \, dx.$$

Elementary calculations will show that the left-hand side becomes

$$\int_U (x \cdot Du)(-\Delta) u \, dx = \frac{2-n}{n} \int_U |Du|^2 \, dx - \frac{1}{2} \int_{\partial U} |Du|^2 (x \cdot \nu) \, dS.$$

Likewise, we calculate that the right-hand term becomes

$$\begin{aligned} \int_U |u|^{p-1} u (x \cdot Du) \, dx &= \frac{1}{p+1} \int_U x \cdot D|u|^{p+1} \, dx \\ &= -\frac{n}{p+1} \int_U |u|^{p+1} \, dx + \frac{1}{p+1} \int_{\partial U} |u|^{p+1} (x \cdot \nu) \, dS \\ &= -\frac{n}{p+1} \int_U |u|^{p+1} \, dx. \end{aligned}$$

The identity follows immediately. □

Proof of Theorem 2.27. Assume otherwise; that is, u is a non-trivial solution of (2.62). If we multiply the PDE by u then integrate over U, we obtain

$$\int_U -u \Delta u \, dx = \int_U |u|^{p+1} \, dx.$$

Then, integration by parts and the zero boundary condition imply that

$$\int_U -u \Delta u \, dx = -\int_{\partial U} u \frac{\partial u}{\partial \nu} \, dS + \int_U |Du|^2 \, dx = \int_U |Du|^2 \, dx.$$

Hence, we arrive at

$$\int_U |u|^{p+1} \, dx = \int_U |Du|^2 \, dx.$$

Inserting this into identity (2.63), we get

$$\left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_U |u|^{p+1} \, dx = \frac{1}{2} \int_{\partial U} |Du|^2 (x \cdot \nu) \, dS \geq 0. \tag{2.64}$$

The inequality on the right is due to $x \cdot \nu \geq 0$ on ∂U, since U is star-shaped with respect to the origin. But this implies that $p \leq (n+2)/(n-2)$, which is a contradiction. □

For special domains and in the setting of positive solutions, this non-existence result can be improved to include the critical exponent $p = (n+2)/(n-2)$. For instance, if $U = B_R(0)$ is the ball of radius $R > 0$ centered at the origin, then $x \cdot \nu = R > 0$ on $\partial B_R(0)$. In view of this and Hopf's lemma, if we take $p \geq (n+2)/(n-2)$, then the inequality in (2.64) becomes a strict one. Thus, we can deduce that $p < (n+2)/(n-2)$ and get a contradiction. Hence, we have the following sharp existence result.

Theorem 2.28. *Let $U = B_R(0)$ for any $R > 0$ and $p > 1$. Then equation* (2.34) *admits a positive classical solution if and only if $p < (n+2)/(n-2)$.*

Interestingly, if $U = \mathbb{R}^n$, then the role of the exponent p reverses in the Lane-Emden equation. Particularly, there holds the following sharp existence result.

Theorem 2.29. *Let $p > 1$ and consider the Lane-Emden equation in the whole space*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{2.65}$$

Then

(a) Equation (2.65) *admits a positive classical solution whenever $p \geq (n+2)/(n-2)$.*

(b) In particular, if $p = (n+2)/(n-2)$, every positive classical solution is radially symmetric and monotone decreasing about some point. Therefore, each positive solution must assume the form

$$u(x) = c_n \Big(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \Big)^{\frac{n-2}{2}}$$

for some constants $c_n, \lambda > 0$ and some point $x_0 \in \mathbb{R}^n$.

(c) Equation (2.65) *has no positive classical solution in the subcritical case, $p < (n+2)/(n-2)$. That is, $u \equiv 0$ is the only non-negative solution of* (2.65).

Proof. In the critical case, the existence of solutions may follow from standard variational methods. In either the super-critical or critical case, the existence of solutions, radially symmetric solutions in particular, follows from a shooting method for ODEs (for a more recent approach combining Brouwer topological fixed point arguments with shooting methods, the reader is referred to [14, 15, 23]). The reason for requiring a shooting method approach is due to the fact that solutions in the super-critical case no longer have finite-energy or belong to a suitable L^p space. Thus, traditional variational methods may no longer apply in this case. Parts (b) and (c) follow from the method of moving planes (see Chapter 5).

□

Consider the more general nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1}u & \text{in } U, \\ \quad u = 0 & \text{on } \partial U. \end{cases} \tag{2.66}$$

We have the following non-existence result, which also follows from a Rellich-Pohozaev type identity. We only state the result and omit the proof (but see [20] for the details).

Theorem 2.30. *Let $u \in C^2(U) \cap C^1(\bar{U})$ be a solution of (2.66), $U \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary, and further assume U is a star-shaped domain with respect to the origin.*

(a) If $\lambda < 0$ and $p \geq (n+2)/(n-2)$,

(b) or if $\lambda \leq 0$ and $p > (n+2)/(n-2)$,

then $u \equiv 0$.

To address the question of existence, particularly that of positive solutions, let λ_1 be the first eigenvalue of the Laplace operator $-\Delta$ on $H_0^1(U)$. Recall λ_1 is positive and characterized by the variational formula (see Theorem 2.11)

$$\lambda_1 = \inf_{u \in H_0^1(U),\, u \neq 0} \frac{\displaystyle\int_U |Du|^2\, dx}{\displaystyle\int_U |u|^2\, dx}. \tag{2.67}$$

The next theorem shows that the previous non-existence result is sharp for $\lambda < 0$. In fact, the existence result remains true for non-negative λ so long as it remains below λ_1.

Theorem 2.31. *Let $1 < p < (n+2)/(n-2)$ and suppose $U \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary. Then there exists a positive solution $u \in H_0^1(U)$ to (2.66) provided that $\lambda < \lambda_1$.*

Proof. Consider the functional

$$E(u) = \frac{1}{2} \int_U |Du|^2 - \lambda |u|^2\, dx. \tag{2.68}$$

It suffices to establish the existence of a minimizer for the functional $E(\cdot)$ over the admissible set

$$M = \{u \in H_0^1(U) \mid \|u\|_{L^{p+1}(U)} = 1\}.$$

The proof is the same as that of Theorem 2.21 except that the boundedness from below and the coercivity of the functional need to be verified. Indeed, this is obvious if $\lambda \leq 0$. Generally, however, we can easily check that (2.67) implies that

$$E(u) \geq \frac{1}{2} \min\left\{1, 1 + \lambda/\lambda_1\right\} \|u\|_{H_0^1(U)} \ \text{ for } \ u \in H_0^1(U), \ \text{ whenever } \lambda < \lambda_1.$$

This shows that $E(\cdot)$ is bounded from below and coercive on $H_0^1(U)$. This completes the proof. $\square$

The previous existence result for positive solutions can be further refined in the critical case. The following is referred to as the Brezis-Nirenberg theorem, and we state it without proof.

Theorem 2.32 (Brezis-Nirenberg)**.** *Let $p = (n+2)/(n-2)$ and suppose $U \subset \mathbb{R}^n$ is a bounded open domain.*

(a) If $n \geq 4$, there exists a positive solution $u \in H_0^1(U)$ of (2.66) *for any $\lambda \in [0, \lambda_1]$.*

(b) If $n = 3$, there exists $\lambda_* \in [0, \lambda_1)$ such that (2.66) *admits a positive solution $u \in H_0^1(U)$ for each $\lambda \in (\lambda_*, \lambda_1)$.*

(c) If $n = 3$ and $U = B_1(0) \subset \mathbb{R}^3$, then $\lambda_* = \lambda_1/4$ and for $\lambda \leq \lambda_*$ there is no positive weak solution to (2.66)*.*

Removing the star-shaped condition on the domain can drastically change the existence of solutions to (2.66). For example, instead let U be the annulus $\{x \in \mathbb{R}^n \mid r_1 < |x| < r_2\}$ and consider the Sobolev space of radially symmetric functions

$$H_{0,rad}^1(U) = \{u \in H_0^1(U) \mid u(x) = u(|x|)\}.$$

Since U is an annulus, the key point here is that the embedding $H_{0,rad}^1(U) \hookrightarrow L^{p+1}(U)$ remains compact for all $p > 1$! So we may apply a variational method with constraint or use a mountain pass approach on $E(\cdot)$ within this class of radial functions. Thus, we can show the existence of infinitely-many radially symmetric positive solutions to (2.66) for any $1 < p < \infty$ and $\lambda \in \mathbb{R}$.

Remark 2.19. *In each of the existence results in this section, the assumption that solutions belong to $C^2(U) \cap C^1(\bar{U})$ can be replaced with the weaker assumption that solutions belong to $H_0^1(U)$. This is due to the regularity theory for weak solutions, which we cover in the next chapter.*

CHAPTER 3

Regularity Theory for Second-order Elliptic Equations

This chapter compiles the basic regularity theory for second-order elliptic equations in divergence form. Roughly speaking, we may classify the study of regularity properties of solutions into three main types:

(A) Schauder's approach or the regularity theory for classical solutions

(B) Calderón-Zygmund or L^p theory

(C) Hölder regularity of weak solutions (using both perturbation and iteration approaches)

Our goal is to cover elementary regularity results along with their proofs for each type, but we must prepare some background material beforehand.

3.1 Preliminaries

In this section, we provide a concise treatment of the tools we require in establishing various regularity results for elliptic equations. Namely, we study the weak L^p, BMO and Morrey–Campanato spaces, the Calderón–Zygmund Decomposition, and the Marcinkiewicz interpolation inequalities.

3.1.1 Flattening out the Boudary

We often assume that the boundary of our domain U is smooth in some sense in order to establish regularity estimates at the boundary. Roughly speaking, such assumptions allows us to flatten the boundary locally and treat it much like what we would do in establishing interior regularity estimates. In particular, let U be an open and bounded domain in $\mathbb{R}^n$ and $k \in \{1, 2, 3, \dots\}$.

Definition 3.1. *We say the boundary ∂U is C^k if for each point $x^0 \in \partial U$ there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that, upon relabeling and reorienting the coordinate axes if necessary, we have*

$$U \cap B_r(x^0) = \{x \in B_r(x^0) \,|\, x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

Likewise, we say ∂U is C^∞ if ∂U is C^k for each $k = 1, 2, 3, \dots,$ and we say ∂U is analytic if the mapping γ is analytic.

We often need to change the coordinates near a boundary point of ∂U as to flatten out the boundary. More precisely, fix $x^0 \in \partial U$ and choose γ and r as in the previous definition. Define $y_i = x_i =: \Phi^i(x)$ if $i = 1, 2, \dots, n-1$ and $y_n = x_n - \gamma(x_1, \dots, x_{n-1}) =: \Phi^n(x)$, and write

$$y = \Phi(x).$$

Similarly, we set $x_i = y_i =: \Psi^i(y)$ for $i = 1, 2, \dots, n-1$ and $x_n = y_n + \gamma(y_1, \dots, y_{n-1}) =: \Psi^n(y)$, and write

$$x = \Psi(y).$$

Then

$$\Phi = \Psi^{-1}$$

and the mapping $x \mapsto \Phi(x) = y$ "straightens out" the boundary ∂U near x^0. Observe additionally that these maps are volume preserving, i.e.,

$$\det D\Phi = \det D\Psi = 1.$$

3.1.2 Weak Lebesgue Spaces and Lorentz Spaces

Let X, or more precisely $(X, \mathcal{A}, \mu)$, be a measure space where μ is a positive, not necessarily finite, measure on X. In most cases, we take $X = \mathbb{R}^n$ with the usual n-dimensional Lebesgue measure. For a measurable function f on X, the **distribution function** of f is the function d_f defined on $[0, \infty)$ as follows:

$$d_f(t) = \mu(\{x \in X : |f(x)| > t\}).$$

Some basic properties of distribution functions are given by the following proposition.

Proposition 3.1. *Let f and g be measurable functions on X. Then for all $s, t > 0$ we have*

(a) $|g| \leq |f|$ *μ–a.e. implies that* $d_g \leq d_f$*,*

(b) $d_{cf}(t) = d_f(t/|c|)$ *for all* $c \in \mathbb{C} \backslash \{0\}$*,*

(c) $d_{f+g}(s+t) \le d_f(s) + d_g(t)$,

(d) $d_{fg}(st) \le d_f(s) + d_g(t)$.

Now we describe L^p norm in terms of the distribution function and define the weak L^p space.

Proposition 3.2. *For $f \in L^p(X)$, $0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty t^{p-1} d_f(t)\, dt.$$

Proof.

$$\begin{aligned} p \int_0^\infty t^{p-1} d_f(t)\, dt &= p \int_0^\infty t^{p-1} \int_X \chi_{\{x \in X : |f(x)| > t\}}\, d\mu(x)\, dt \\ &= \int_X \int_0^{|f(x)|} p t^{p-1}\, dt\, d\mu(x) \\ &= \int_X |f(x)|^p\, d\mu(x) \\ &= \|f\|_{L^p}^p, \end{aligned}$$

where we used Fubini's Theorem in the second equality. □

Definition 3.2. *For $0 < p < \infty$, the space weak $L^p(X)$, also denoted by $L_w^p(X)$ or $L^{p,\infty}(X)$, is defined as the set of all μ-measurable functions f such that*

$$\begin{aligned} \|f\|_{L^{p,\infty}} &= \inf \left\{ C > 0 : d_f(t) \le \left(\frac{C}{t} \right)^p \quad \text{for all} \quad t > 0 \right\} \\ &= \sup \left\{ t d_f(t)^{1/p} : t > 0 \right\} \end{aligned}$$

is finite. The space weak $L^\infty(X)$ is by definition $L^\infty(X)$.

Remark 3.1. *The weak $L^p(X)$ space is commonly denoted by $L_w^p(X)$ or by its equivalent Lorentz space characterization $L^{p,\infty}(X)$. Moreover, we can show that*

(a) $\|f\|_{L^{p,\infty}} = 0 \Longrightarrow f = 0$ *μ a.e.,*

(b) $\|kf\|_{L^{p,\infty}} = |k| \|f\|_{L^{p,\infty}}$,

(c) $\|f+g\|_{L^{p,\infty}} \le \max\{2, 2^{1/p}\}(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$.

Hence, the triangle inequality does not hold so that $L^{p,\infty}(X)$ is a quasi-normed linear space for $0 < p < \infty$. In fact, these spaces are complete.

Obviously, the weak L^p spaces are larger than L^p spaces.

Proposition 3.3. *For any* $0 < p < \infty$ *and any* $f \in L^p(X)$, *we have*

$$\|f\|_{L^{p,\infty}} \leq \|f\|_{L^p}.$$

Hence, $L^p(X) \hookrightarrow L^{p,\infty}(X)$.

Proof. This is a trivial consequence of Chebyshev's inequality:

$$t^p d_f(t) \leq \int_{\{x \in X \,:\, |f(x)| > t\}} |f(x)|^p \, d\mu(x).$$

□

Definition 3.3. *An operator* $T : L^p(X) \longrightarrow L^q(X)$ *is of* **strong type** (p, q) *if*

$$\|Tf\|_{L^q} \leq C\|f\|_{L^p} \quad \textit{for all} \quad f \in L^p(X).$$

Similarly, T *is of* **weak type** (p, q) *if*

$$\|Tf\|_{L^{q,\infty}} \leq C\|f\|_{L^p} \quad \textit{for all} \quad f \in L^p(X).$$

For completeness, we introduce the Lorentz spaces in which the Lebesgue and weak Lebesgue spaces are special cases. First, if f is a real (or comlex) valued function defined on X, then the **decreasing rearrangement** of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s > 0 \,|\, d_f(s) \leq t\}.$$

We adopt the convention that $\inf \emptyset = \infty$, thus $f^*(t) = \infty$ whenever $d_f(s) > t$ for all $s \geq 0$. Now, given a measurable function f on X and $0 < p, q \leq \infty$, define

$$\|f\|_{L^{p,q}(X)} = \Big(\int_0^\infty (t^{1/p} f^*(t))^q \, \frac{dt}{t} \Big)^{1/q}$$

whenever $q < \infty$, and if $q = \infty$ we take

$$\|f\|_{L^{p,\infty}(X)} = \sup_{t > 0} t^{1/p} f^*(t).$$

Then the set of all f with $\|f\|_{L^{p,q}(X)} < \infty$ is denoted by $L^{p,q}(X)$ and is called the **Lorentz space** with indices p and q. It is interesting to note several properties of the decreasing rearrangement of f. Namely, we have that

(a) $d_f = d_{f^*}$,

(b) $(|f|^p)^* = (f^*)^p$ whenever $0 < p < \infty$,

(c) $\int_X |f|^p \, d\mu = \int_0^\infty f^*(t)^p \, dt$ whenever $0 < p < \infty$,

(d) $\sup_{t>0} t^q f^*(t) = \sup_{\alpha > 0} \alpha (d_f(\alpha))^q$ for $0 < q < \infty$.

In view of these properties, it is simple to verify that $L^{p,p}(X) = L^p(X)$, $L^{\infty,\infty}(X) = L^\infty(X)$, and weak $L^p(X) = L^{p,\infty}(X)$.

3.1.3 The Marcinkiewicz Interpolation Inequalities

The following is known as the Marcinkiewicz interpolation theorem. A more general "non-diagonal" version involving the Lorentz spaces holds, but we shall not make use of it in these notes and thus omit it.

Theorem 3.1 (Marcinkiewicz interpolation). *Let T be a linear operator from $L^p(X) \cap L^q(X)$ into itself with $1 \leq p < q \leq \infty$. If T is of weak type (p,p) and weak type (q,q), then for any $p < r < q$, T is of strong type (r,r). More precisely, if there exist constants B_p and B_q such that*

$$d_{Tf}(t) \leq \left(\frac{B_p \|f\|_p}{t}\right)^p$$

and

$$d_{Tf}(t) \leq \left(\frac{B_q \|f\|_q}{t}\right)^q$$

for all $f \in L^p(X) \cap L^q(X)$, then

$$\|Tf\|_{L^r} \leq C B_p^{\theta} B_q^{1-\theta} \|f\|_r \quad \text{for all} \quad f \in L^p(X) \cap L^q(X),$$

where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

and $C = C(p,q,r)$ is a positive constant. Note that if $q = \infty$, then the $L^q(X)$ and $L^{q,\infty}(X)$ spaces and their norms above are replaced with the space $L^\infty(X) = L^{\infty,\infty}(X)$ and its norm.

3.1.4 Calderón–Zygmund and the John-Nirenberg Lemmas

Lemma 3.1 (Calderón–Zygmund Decomposition). *For $f \in L^1(\mathbb{R}^n)$, a fixed $\alpha > 0$, there exists E and G such that*

(a) $\mathbb{R}^n = E \cup G$, $E \cap G = \emptyset$,

(b) $|f(x)| \leq \alpha$ a.e. $x \in E$,

(c) $G = \cup_{k=1}^{\infty} Q_k$, $\{Q_k\}$ are disjoint cubes for which

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} |f(x)|\, dx \leq 2^n \alpha.$$

Lemma 3.2 (John-Nirenberg). *Suppose $u \in L^1(U)$ satisfies*

$$\int_{B_r(x)} |u - (u)_{x,r}|\, dy \leq M r^n \quad \text{for any} \quad B_r(x) \subset U.$$

Then there holds for any $B_r(x) \subset U$

$$\int_{B_r(x)} e^{\frac{p_0}{M}|u-(u)_{x,r}|}\, dy \leq C r^n$$

for some positive p_0 and C depending only on n.

3.1.5 L^p Boundedness of Integral Operators

We briefly introduce some basic results on integral operators of convolution type but our goal is to ultimately prove the Hardy-Littlewood-Sobolev (HLS) inequality. However, we will need some basic properties of the Hardy-Littlewood maximal function in order to prove the HLS inequality. The weak Lebesgue spaces, the Calderón–Zygmund decomposition and the Marcinkiewicz interpolation inequalities will play very important roles here.

Specifically, the operators we consider are examples of singular integral operators whose kernels do not belong to a proper L^p space but rather to a weak L^p space, e.g., the Riesz type kernel $|x|^{-(n-\alpha)}$ belongs to $L^{p,\infty}(\mathbb{R}^n)$ but not to $L^p(\mathbb{R}^n)$ when $p = n/(n-\alpha)$. This type of issue is relevant in the L^p regularity theory for elliptic partial differential equations studied later in this chapter. Particularly, we shall see in Section 3.2 that deriving $W^{2,p}$ a priori estimates on weak solutions requires showing certain differential operators involving the Newtonian potentials are weak and strong type operators. A similar dichotomy appears for the maximal function operators.

The function

$$\mathcal{M}(f)(x) = \sup_{\delta>0} Avg_{B_\delta(x)}|f| = \sup_{\delta>0} \frac{1}{n\omega_n \delta^n} \int_{B_\delta(0)} |f(x-y)|\, dy$$

is called the **centered Hardy-Littlewood maximal function** of f. Likewise, the function

$$M(f)(x) = \sup_{\delta>0,\, |y-x|<\delta} Avg_{B_\delta(y)}|f|$$

is called the **uncentered Hardy-Littlewood maximal function** of f.

Clearly, $\mathcal{M}(f) \leq M(f)$. Also, $\mathcal{M}(f) = \mathcal{M}(|f|) \geq 0$, i.e., the maximal function is a positive operator, and obviously $\mathcal{M}$ maps L^∞ to itself, i.e.,

$$\|\mathcal{M}(f)\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

We show that the maximal function as an integral operator is of weak type $(1,1)$ and thus is of strong type (p,p) for any $1 < p < \infty$ by interpolation. The proof of this requires the following basic result which is sometimes referred to as the Vitali covering lemma.

Lemma 3.3. *Let $\{B_1, B_2, \ldots, B_k\}$ be a finite collection of open balls in $\mathbb{R}^n$. Then there exists a finite subcollection $\{B_{j_1}, B_{j_2}, \ldots, B_{j_\ell}\}$ of pairwise disjoint balls such that*

$$\sum_{r=1}^{\ell} |B_{j_r}| \geq 3^{-n} \Big| \bigcup_{i=1}^{k} B_i \Big|. \tag{3.1}$$

Proof. Without loss of generality, we can assume that the collection of balls satisfies

$$|B_1| \geq |B_2| \geq \ldots \geq |B_k|.$$

Let $j_1 = 1$. Having chosen $j_1, j_2, \ldots, j_i$, let j_{i+1} be the least index $s > j_i$ such that $\cup_{m=1}^{i} B_{j_m}$ is disjoint from B_s. Since we have a finite collection of balls, this process must stop after some ℓ finite number of steps. Indeed, this yields a finite subcollection of pairwise disjoint balls $B_{j_1}, B_{j_2}, \ldots, B_{j_\ell}$. If some B_m was not selected, i.e., $m \notin \{j_1, j_2, \ldots, j_\ell\}$, then B_m must intersect a selected ball B_{j_r} for some $j_r < m$. Then B_m has smaller size than B_{j_r} and we must have $B_m \subseteq 3B_{j_r}$. This shows that the union of the unselected balls is contained in the union of triples of the selected balls. Thus, the union of all balls is contained in the union of the triples of the selected balls and so

$$\Big|\bigcup_{i=1}^{k} B_i\Big| \leq |\bigcup_{r=1}^{\ell} 3B_{j_r}| \leq \sum_{r=1}^{\ell} |3B_{j_r}| = 3^n \sum_{r=1}^{\ell} |B_{j_r}|.$$

This completes the proof. $\square$

Theorem 3.2. *The uncentered Hardy-Littlewood maximal function maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with constant at most 3^n and also $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$ with constant at most $3^{n/p}p(p-1)^{-1}$. The same is true for the centered maximal operator $\mathcal{M}$.*

Proof. Since $M(f) \geq \mathcal{M}(f)$, we have

$$\{x \in \mathbb{R}^n \,|\, |\mathcal{M}(f)(x)| > t\} \subseteq \{x \in \mathbb{R}^n \,|\, |M(f)(x)| > t\},$$

and therefore it suffices to show that

$$d_{M(f)}(t) := |\{x \in \mathbb{R}^n \,|\, |M(f)(x)| > t\}| \leq 3^n \frac{\|f\|_{L^1(\mathbb{R}^n)}}{t}. \tag{3.2}$$

Step 1: We claim that the set

$$E_t = \{x \in \mathbb{R}^n \,|\, |M(f)(x)| > t\}$$

is open. Indeed, for $x \in E_t$ there is an open ball B_x containing x such that the average of $|f|$ over B_x is strictly bigger than t. Then the uncentered maximal function of any other point in B_x is also bigger than t, and thus B_x is contained in E_t. This proves that E_t is open.

Step 2: Estimate (3.2) holds.

Let K be any compact subset of E_t. For each $x \in K$ there exists an open ball B_x containing the point x such that

$$\int_{B_x} |f(y)|\, dy > t|B_x|. \tag{3.3}$$

Observe that $B_x \subset E_t$ for all x, and by compactness there exists a finite subcover

$$\{B_{x_1}, B_{x_2}, \ldots, B_{x_k}\} \text{ of the subset } K.$$

In view of Lemma 3.3, we find a subcollection of pairwise disjoint balls $B_{x_{j_1}}, \ldots, B_{x_{j_\ell}}$ such that (3.1) holds and combining this with (3.3) yields

$$|K| \leq \Big| \bigcup_{i=1}^{k} B_{x_i} \Big| \leq 3^n \sum_{i=1}^{\ell} |B_{x_{j_i}}| \leq \frac{3^n}{t} \sum_{i=1}^{\ell} \int_{B_{x_{j_i}}} |f(y)|\, dy \leq \frac{3^n}{t} \int_{E_t} |f(y)|\, dy$$

since all the balls $B_{x_{j_i}}$ are disjoint and contained in E_t. From this we deduce (3.2) after taking the supremum over all compact subsets of $K \subseteq E_t$ and using the inner regularity of the Lebesgue measure. This verifies $M = M(f)$ (as well as $\mathcal{M} = \mathcal{M}(f)$) is of weak type $(1,1)$. Recall that M is of strong type (p,p) with $p = \infty$. Thus, the Marcinkiewicz interpolation theorem (see Theorem 3.1) implies the operator M is of strong type (p,p) for all $1 < p < \infty$. This completes the proof of the theorem. $\square$

The following result states that the maximal operator controls the averages of a function with respect to any radially decreasing integrable function. We omit the proof but refer to Theorem 2.1.10 in [12].

Theorem 3.3. *Let $k \geq 0$ be a function on $[0,\infty)$ that is continuous except at a finite number of points. Suppose that $K(x) = k(|x|)$ is an integrable function on $\mathbb{R}^n$ and satisfies*

$$K(x) \geq K(y) \;\; \text{whenever} \;\; |x| \leq |y|,$$

i.e., k is decreasing. Then

$$\sup_{\epsilon>0} |f| * K_\epsilon(x) \leq \|K\|_{L^1(\mathbb{R}^n)} \mathcal{M}(f)(x)$$

for all locally integrable functions f on $\mathbb{R}^n$. Here $K_\epsilon(x) = \epsilon^{-n} K(x/\epsilon)$. An important case is when $K(x) = |x|^{\alpha-n}\chi_{|x|<R}(x)$ for any fixed $R \in (0,\infty)$ and $\alpha \in (0,n)$.

With the results presented above, we are now ready to offer some important applications of the Hardy-Littlewood maximal functions.

The Lebesgue Differentiation Theorem

Theorem 3.4 (Lebesgue Differentiation Theorem)**.** *For any $f \in L^1_{loc}(\mathbb{R}^n)$, there holds*

$$\lim_{r \longrightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy = f(x) \;\; \text{for a.e.} \;\; x \in \mathbb{R}^n. \tag{3.4}$$

Consequently, $|f| \leq \mathcal{M}(f)$ a.e.

Before we prove this, we need some preliminary tools. First, let (X,μ) and (Y,ν) be two measure spaces, $p \in (0,\infty]$ and $q \in (0,\infty)$. Suppose that D is a dense subspace of $L^p(X,\mu)$ and for every $\epsilon > 0$, T_ϵ is a linear operator on $L^p(X,\mu)$ with values in the set of measurable functions on Y. Define the sublinear operator

$$T_*(f)(x) = \sup_{\epsilon>0} |T_\epsilon(f)(x)|.$$

Theorem 3.5. *Let $p, q \in (0, \infty)$. Suppose that for some constant $C > 0$ and all $f \in L^p(X, \mu)$ we have*

$$\|T_*(f)\|_{L^{q,\infty}} \leq C\|f\|_{L^p},$$

and for all $f \in D$,

$$\lim_{\epsilon \longrightarrow 0} T_\epsilon(f) = T(f) \tag{3.5}$$

exists and is finite for ν-a.e. and defines a linear operator on D. Then, for all $f \in L^p(X, \mu)$, the limit (3.5) exists and finite ν-a.e. and uniquely defines an operator T on $L^p(X, \mu)$, by the continuous extension of T on the dense subspace D, such that

$$\|T(f)\|_{L^{q,\infty}} \leq C\|f\|_{L^p}. \tag{3.6}$$

Proof. Given $f \in L^p(X, \mu)$, we define the oscillation of f by

$$O_f(y) = \limsup_{\epsilon \to 0} \limsup_{\theta \to 0} |T_\epsilon(f)(y) - T_\theta(f)(y)|.$$

We claim that for all $f \in L^p(X, \mu)$ and $\delta > 0$,

$$\nu(\{y \in Y \mid O_f(y) > \delta\}) = 0. \tag{3.7}$$

Once, we prove this claim, then $O_f(y) = 0$ for ν-a.e. y, which further implies that $T_\epsilon(f)(y)$ is Cauchy for ν-a.e. y. This implies that $T_\epsilon(f)(y)$ converges ν-a.e. to some $T(f)(y)$ as $\epsilon \longrightarrow 0$. The operator T defined this way on $L^p(X, \mu)$ is linear and extends T defined on D.

We now prove the claim. Choose $\eta > 0$ and by density, we may choose $g \in D$ such that $\|f - g\|_{L^p} < \eta$. Since $T_\epsilon(g) \longrightarrow T(g)$ ν-a.e., it follows that $O_g = 0$ ν-a.e. From this and the linearity of T_ϵ, we conclude that

$$O_f(y) \leq O_g(y) + O_{f-g}(y) = O_{f-g}(y) \text{ for } \nu\text{-a.e. } y.$$

Now for any $\delta > 0$, we have

$$\begin{aligned} \nu(\{y \in Y \mid O_f(y) > \delta\}) &\leq \nu(\{y \in Y \mid O_{f-g}(y) > \delta\}) \\ &\leq \nu(\{y \in Y \mid 2T_*(f-g)(y) > \delta\}) \\ &\leq ((2C/\delta)\|f - g\|_{L^p})^q \leq (2C\eta/\delta)^q. \end{aligned}$$

Then sending $\eta \longrightarrow 0$, we deduce (3.7). We thus conclude that $T_\epsilon(f)$ is a Cauchy sequence and hence converges ν-a.e. to some $T(f)$. Since $|T(f)| \leq |T_*(f)|$, the estimate (3.6) follows immediately. □

Proof of Theorem 3.4. Since $\mathbb{R}^n$ is locally compact and is the union of the open balls $B_N(0)$, $N = 1, 2, 3, \ldots$, it suffices to prove the theorem for almost every x inside the ball $B_N(0)$. Then we may take f supported in a larger ball, thus working with f integrable over the whole space $\mathbb{R}^n$.

Let $T_\epsilon(f) = K_\epsilon * f$, where $K_\epsilon(x) = \epsilon^{-n}k(x/\epsilon)$ with $k = |B_1(0)|^{-1}\chi_{B_1(0)}$. We know that the corresponding operator T_* is controlled by the centered Hardy-Littlewood maximal function $\mathcal{M}$ (see Theorem 3.3), which maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, i.e., $\mathcal{M}$ is an operator of weak type $(1,1)$. Hence, T_* must also be of weak type $(1,1)$.

It is easy to show that (3.4) holds in the space of continuous functions f with compact support, which is dense in $L^1(\mathbb{R}^n)$. From this and the fact that T_* maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, Theorem 3.5 implies that (3.4) holds for all $f \in L^1(\mathbb{R}^n)$. □

The Hardy-Littlewood-Sobolev inequality

Consider the integral operator

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

and recall its boundedness property in L^p spaces, which we introduced earlier as the Hardy-Littlewood-Sobolev (HLS) inequality. The proof that we present here center on the strong boundedness of the Hardy-Littlewood maximal function and Theorem 3.3.

Theorem 3.6 (HLS inequality)**.** *Let $\alpha \in (0,n)$ and $1 < p < q < \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Then there exists a finite positive constant $C = C(n,\alpha,p)$ such that for all $f \in L^p(\mathbb{R}^n)$ there holds

$$\|I_\alpha(f)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.8}$$

Proof. The main idea is to estimate the operator I_α in terms of the Hardy-Littlewood maximal function. Specifically, our estimates below will involve the uncentered maximal operator $M(f)$. First, observe that $I_\alpha(f)$ is well-defined in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ which is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. So it suffices to assume that $f \in \mathcal{S}(\mathbb{R}^n)$. We may also assume that $f \ge 0$ since $I_\alpha(|f|) \ge |I_\alpha(f)|$. Now consider the splitting,

$$\int_{\mathbb{R}^n} f(x-y)|y|^{\alpha-n}\, dy = J_1(f)(x) + J_2(f)(x),$$

where

$$J_1(f)(x) = \int_{|y|<R} f(x-y)|y|^{\alpha-n}\, dy,$$
$$J_2(f)(x) = \int_{|y|\ge R} f(x-y)|y|^{\alpha-n}\, dy,$$

and $R > 0$ is some constant to be specified below.

Estimating J_1: Particularly, J_1 is given by convolution with the function $|y|^{\alpha-n}\chi_{|y|<R}$. So by applying Theorem 3.3, we have that

$$J_1(f)(x) \leq M(f)(x) \int_{|y|<R} |y|^{\alpha-n}\, dy = \frac{\omega_n}{\alpha} R^\alpha M(f)(x).$$

Estimating J_2: Hölder's inequality yields

$$\begin{aligned} |J_2(f)(x)| &\leq \Big(\int_{|y|\geq R} |y|^{p(\alpha-n)/(p-1)}\, dy \Big)^{(p-1)/p} \|f\|_{L^p(\mathbb{R}^n)} \\ &= \Big(\frac{(p-1)q\omega_n}{pn} \Big)^{(p-1)/p} R^{-n/q} \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Combining the above estimates for J_1 and J_2 yields for any $R > 0$

$$I_\alpha(f)(x) \leq C(n,\alpha,p)(R^\alpha M(f)(x) + R^{-n/q}\|f\|_{L^p(\mathbb{R}^n)}).$$

Hence, by choosing a constant multiple of the quantity

$$R = \|f\|_{L^p(\mathbb{R}^n)}^{p/n} (M(f)(x))^{-p/n},$$

we reduce the previous estimate to

$$I_\alpha(f)(x) \leq C(n,\alpha,p) M(f)(x)^{p/q} \|f\|_{L^p(\mathbb{R}^n)}^{1-p/q}. \tag{3.9}$$

We deduce the desired result by raising estimate (3.9) to the power q then integrating over $\mathbb{R}^n$ then using the fact that $M(f)$ is of strong type (p,p) for any $1 < p < \infty$ (see Theorem 3.2). This completes the proof. □

Remark 3.2. *Interestingly enough, a weaker version of the HLS inequality holds in the endpoint case $p = 1$ but with the original estimate* (3.8) *being replaced with the estimate*

$$\|I_\alpha(f)\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C(n,\alpha)\|f\|_{L^1(\mathbb{R}^n)}$$

where $q = n/(n-\alpha)$. The proof of this is just as before since the weaker inequality will follow from the estimate (3.9) *and the fact that $M(f)$ is of weak type* $(1,1)$.

The Hilbert and Riesz Transforms

For completeness, we look at another prototypical example of a singular integral operator of convolution type called the Hilbert transform. There are several ways to define the Hilbert transform. First, we give its definition as a convolution operator with a certain principle value distribution. We begin by defining the distribution $W_0 \in \mathcal{S}'(\mathbb{R})$ as

$$\langle W_0, \varphi\rangle = \pi^{-1} \lim_{\epsilon\to 0} \int_{\epsilon\leq|x|\leq 1} \frac{\varphi(x)}{x}\, dx + \pi^{-1} \int_{|x|\geq 1} \frac{\varphi(x)}{x}\, dx \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}).$$

Then the Hilbert transform of $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$H(f)(x) = (W_0 * f)(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(x-y)}{y}\, dy = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy, \tag{3.10}$$

where

$$P.V. \int_{-\infty}^{\infty} F(x,y)\, dy = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} F(x,y)\, dy$$

is the usual principle value integral.

Remark 3.3. *Note that*

$$\int_{-\infty}^{\infty} \frac{f(x-y)}{y}\, dy$$

does not converge absolutely, and it is important to notice that the function $1/y$ integrated over $[-1,-\epsilon] \cap [\epsilon, 1]$ has mean value 0. Therefore, this is precisely why we must treat the above improper integral in the principal value sense. Also, for each $x \in \mathbb{R}$, $H(f)(x)$ is defined for all integrable functions f on $\mathbb{R}$ that satisfy a Hölder condition near the point x.

Alternatively, we can define the Hilbert transform using the Fourier transform. Namely, there holds

$$\widehat{W_0}(\xi) = -isgn(\xi),$$

and so

$$H(f)(x) = \mathcal{F}^{-1}(\widehat{f}(\xi)[-isgn(\xi)])(x). \tag{3.11}$$

An immediate consequence of (3.11) is that H is an isometry on $L^2(\mathbb{R})$, i.e.,

$$\|H(f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Moreover, it follows that the adjoint of H is $H^* = -H$. Now, as with the Hardy-Littlewood maximal operator, the Hilbert transform is of strong type (p,p) for all $1 < p < \infty$. We sketch the proof of this. First, we can show the estimate

$$|\{x \in \mathbb{R} \,|\, |H(\chi_E)(x)| > t\}| \leq \frac{2}{\pi} \frac{|E|}{t}, \quad t > 0,$$

holds for all subsets E of the real line of finite measure. This inequality and a basic result (see Theorem 1.4.19 in [12]) ensure H is bounded on $L^p(\mathbb{R})$ for $1 < p < 2$. By duality, $H^* = -H$ is bounded on $L^p(\mathbb{R})$ for $2 < p < \infty$. Thus, H is also bounded on $L^p(\mathbb{R})$ for $2 < p < \infty$. Finally, H is an isometry on $L^2(\mathbb{R})$. This completes the proof.

The Riesz tranforms are the n-dimensional analogue of the Hilbert transform. To introduce such transforms, we introduce the tempered distributions W_j on $\mathbb{R}^n$, for $1 \leq j \leq n$ as follows. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let

$$\langle W_j, \varphi \rangle = \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+1}{2}}} P.V. \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} \varphi(y)\, dy.$$

Then the **jth Riesz transform** of f, denoted by $R_j(f)$, is given by convolution with W_j, i.e.,

$$R_j(f)(x) = (f * W_j)(x) = \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+1}{2}}} P.V. \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Alternatively, the jth Riesz transform can be defined via the Fourier transform, i.e.,

$$R_j(f)(x) = \mathcal{F}^{-1}(-\frac{i\xi_j}{|\xi|}\widehat{f}(\xi))(x) \text{ for all } f \in \mathcal{S}(\mathbb{R}^n).$$

Interestingly enough, the Riesz transforms satisfy

$$-Identity = \sum_{j=1}^{n} R_j^2.$$

Likewise, the jth Riesz transforms R_j are bounded operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Application of Riesz tranforms to the Poisson equation

Another interesting application of Riesz tranforms is to Poisson's equation. Namely, suppose that f belongs to $\mathcal{S}(\mathbb{R}^n)$ and u is a tempered distribution that solves the elliptic equation

$$-\Delta u = f.$$

Indeed, there holds from the Fourier transform that

$$(4\pi^2|\xi|^2)\widehat{u}(\xi) = \widehat{f}(\xi).$$

Notice that for all $1 \leq j,k \leq n$ we have

$$\partial_j\partial_k u = \mathcal{F}^{-1}((2\pi i\xi_j)(2\pi i\xi_k)\widehat{u}(\xi)) = \mathcal{F}^{-1}\Big((2\pi i\xi_j)(2\pi i\xi_k)\frac{\widehat{f}(\xi)}{4\pi^2|\xi|^2}\Big) = R_jR_k(f) = R_jR_k(-\Delta u).$$

That is, we conclude that $\partial_j\partial_k u$ are functions. Thus, Riesz transforms provide an explicit way to recover second-order derivatives in terms of the Laplacian.

Remark 3.4. *If $f = 0$, then we reduce the problem to the Laplace equation, $\Delta u = 0$, and a solution $u \in \mathcal{S}'(\mathbb{R}^n)$ is usually called a **harmonic distribution**. As above, applying the Fourier transform yields $\widehat{\Delta u} = 0$ and so*

$$-4\pi^2|\xi|^2\widehat{u} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

This implies that $\widehat{u}$ is supported at the origin, so applying the inverse Fourier transform implies the Liovuille theorem that u is a polynomial.

3.2 $W^{2,p}$ Regularity for Weak Solutions

This section covers the L^p or so-called Calderón-Zygmund regularity theory for second-order elliptic equations.

3.2.1 $W^{2,p}$ A Priori Estimates

Initially, we will establish the $W^{2,p}$ a priori estimates for the Newtonian potentials, then extend the result to general elliptic equations.

Theorem 3.7 ($W^{2,p}$ a priori Estimate for the Newtonian Potential)**.** *Let $f \in L^p(U)$ for $1 < p < \infty$, and let $w = \Gamma * f$ be the Newtonian potential of f. Then $w \in W^{2,p}(U)$ and*

$$-\Delta w = f(x) \quad a.e.\ x \in U \quad and \quad \|D^2 w\|_{L^p} \leq C\|f\|_{L^p}.$$

Proof. We provide a sketch of the proof in four key steps. We define the linear operator T by

$$Tf = D_{ij}\Gamma * f.$$

Observe that it suffices to show that T is a bounded linear operator on $L^p(U)$.
Step 1: $T : L^2(U) \longrightarrow L^2(U)$ is a bounded linear operator, i.e., T is of strong type $(2,2)$.

Let $f \in C_0^\infty(U) \subset C_0^\infty(\mathbb{R}^n)$. Recall that $w \in C^\infty(\mathbb{R}^n)$ and satisfies Poisson's equation

$$-\Delta w = f(x) \ \text{ in } \ \mathbb{R}^n.$$

With the help of the Fourier transform and Plancherel's identity,

$$\begin{aligned}
\int_U |f(x)|^2\, dx &= \int_{\mathbb{R}^n} |f(x)|^2\, dx = \int_{\mathbb{R}^n} |\Delta w|^2\, dx = \int_{\mathbb{R}^n} |\widehat{\Delta w(\xi)}|^2\, d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^4 |\widehat{w(\xi)}|^2\, dx = \sum_{k,j=1}^n \int_{\mathbb{R}^n} \xi_k^2 \xi_j^2 |\widehat{w(\xi)}|^2\, d\xi \\
&= \sum_{k,j=1}^n \int_{\mathbb{R}^n} |\widehat{D_{kj} w(\xi)}|^2\, d\xi = \sum_{k,j=1}^n \int_{\mathbb{R}^n} |D_{kj} w(x)|^2\, dx \\
&= \int_{\mathbb{R}^n} |D^2 w|^2\, dx.
\end{aligned}$$

Hence, $\|Tf\|_{L^2} \leq \|f\|_{L^2}$ for all $f \in C_0^\infty(U)$ and so $T : L^2(U) \longrightarrow L^2(U)$ is a bounded linear operator simply by the density of $C_0^\infty(U)$ in $L^2(U)$.

Step 2: T is of weak type $(1,1)$.

This result follows from the Calderón–Zygmund decomposition and we skip its proof for the sake of brevity, but the reader is referred to [5][page 82] for the proof.

Step 3: T is of strong type (p,p) for any $1 < p < \infty$.

Since T is of weak type $(1,1)$ and is of strong type $(2,2)$—therefore is of weak type $(2,2)$—the Marcinkiewicz interpolation theorem implies that T is of strong type (r,r) for $1 < r \leq 2$. Given any $2 < q < \infty$, let $r = \frac{q}{q-1} \in (1,2]$. By duality and the fact that T is of strong type (r,r), we see that

$$\begin{aligned}
\|Tf\|_{L^q} &= \sup_{\|g\|_{L^r}=1} \langle g, Tf\rangle := \sup_{\|g\|_{L^r}=1} \int_U g(x)Tf(x)\,dx \\
&= \sup_{\|g\|_{L^r}=1} \langle Tg, f\rangle \leq \sup_{\|g\|_{L^r}=1} \|Tg\|_{L^r}\|f\|_{L^q} \\
&\leq \sup_{\|g\|_{L^r}=1} C_r\|g\|_{L^r}\|f\|_{L^q} \\
&\leq C_r\|f\|_{L^q}.
\end{aligned}$$

Thus, T is of strong type (q,q) for $q \in (2,\infty)$. Hence, T is of strong type (p,p) for any $1 < p < \infty$. $\square$

Now we present the $W^{2,p}$ a priori estimates on strong solutions for the uniformly elliptic equation with bounded coefficients:

$$Lu = f(x) \ \text{ in } \ U. \tag{3.12}$$

Definition 3.4. *We say that u is a* **strong solution** *of* (3.12) *if u is twice weakly differentiable in U and satisfies the equation almost everywhere in U.*

Throughout this section, we assume $U \subset \mathbb{R}^n$ is bounded and open with $C^{2,\alpha}$ boundary, $a^{ij} \in C(\bar{U})$, $b^i \in L^q(U)$ and $c \in L^q(U)$ for some $q \in (n,\infty]$. In the details below, we will assume $q = \infty$ for simplicity.

Theorem 3.8 ($W^{2,p}$ Estimates for Uniformly Elliptic Equations)**.** *Let $1 < p < \infty$, $f \in L^p(U)$, and let $u \in W^{2,p}(U) \cap H_0^1(U)$ be a strong solution of* (3.12)*. Then*

$$\|u\|_{W^{2,p}} \leq C\left(\|u\|_{L^p} + \|f\|_{L^p}\right)$$

where $C = C(\lambda, \Lambda, n, p, U, \|b_i\|_{L^\infty}, \|c\|_{L^\infty})$ is a positive constant.

Proof. The proof can be separated into two major estimates—the interior estimate and the boundary estimate.

Part I: Interior Estimate

$$\|D^2u\|_{L^p(K)} \leq C\left(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right) \tag{3.13}$$

where K is any compact subset of U.

Part II: Boundary Estimate

$$\|D^2u\|_{L^p(U\setminus U_\delta)} \leq C\left(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right) \tag{3.14}$$

where $U_\delta = \{x \in U \,|\, dist(x, \partial U) > \delta\}$.

Part III: The interior and boundary estimates imply

$$\|u\|_{W^{2,p}(U)} \leq C\left(\|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right). \tag{3.15}$$

To see this, it is obvious that both estimates yield

$$\begin{aligned}\|u\|_{W^{2,p}(U)} &\leq \|u\|_{W^{2,p}(U\setminus U_{2\delta})} + \|u\|_{W^{2,p}(U_\delta)} \\ &\leq C\left(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right).\end{aligned} \tag{3.16}$$

We have the following estimate

$$\begin{aligned}\|Du\|_{L^p(U)} &\leq C\|u\|^{1/2}_{L^p(U)}\|D^2u\|^{1/2}_{L^p(U)} \\ &\leq \epsilon\|D^2u\|_{L^p(U)} + \frac{C}{4\epsilon}\|u\|_{L^p(U)}\end{aligned}$$

where the first inequality is the well-known Gagliardo–John–Nirenberg interpolation inequality and the second inequality is the basic Cauchy inequality with ϵ. Substituting this into (3.16) yields

$$\|u\|_{W^{2,p}(U)} \leq C\epsilon\|D^2u\|_{L^p(U)} + C\left(\frac{C}{4\epsilon}\|u\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}\right).$$

If we choose $\epsilon < \frac{1}{2C}$, we can absorb the $C\epsilon\|D^2u\|_{L^p(U)}$ term on the right-hand side by the left-hand side and arrive at the desired estimate. □

Let us give provide the details in obtaining interior and boundary estimates.

Part I: Interior Estimates We proceed using the well-known method of frozen coefficients. Define the cut-off function $\varphi \in C_c^\infty(\mathbb{R})$ to be the function

$$\varphi(s) := \begin{cases} 1 & \text{if } s \leq 1, \\ 0 & \text{if } s \geq 2. \end{cases}$$

Then we measure the *module* continuity of the coefficients a^{ij} with

$$\epsilon(\delta) = \sup_{|x-y|\leq\delta, x,y\in U, 1\leq i,j\leq n} |a^{ij}(x) - a^{ij}(y)|.$$

Note that the function $\epsilon(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. Then for any $x_0 \in U_{2\delta}$, let

$$\eta(x) = \varphi\left(\frac{|x - x_0|}{\delta}\right) \quad \text{and} \quad w(x) = \eta(x)u(x).$$

We compute

$$
\begin{aligned}
a^{ij}(x_0)\frac{\partial^2 w}{\partial x_i \partial x_j} &= (a^{ij}(x_0) - a^{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} \\
&= (a^{ij}(x_0) - a^{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + \eta(x)a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} \\
&\quad + a^{ij}(x)u(x)\frac{\partial^2 \eta}{\partial x_i \partial x_j} + 2a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_j} \\
&= (a^{ij}(x_0) - a^{ij}(x))\frac{\partial^2 w}{\partial x_i \partial x_j} + \eta(x)\left(b^i(x)\frac{\partial u}{\partial x_i} + c(x)u - f(x)\right) \\
&\quad + a^{ij}(x)u(x)\frac{\partial^2 \eta}{\partial x_i \partial x_j} + 2a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_j} \\
&:= F(x) \quad \text{for} \quad x \in \mathbb{R}^n.
\end{aligned}
$$

Notice that all terms in F are supported in $B_{2\delta}(x_0) \subset U$. By the uniformly elliptic condition, we can assume $a^{ij}(x_0) = \delta_{ij}$ by a simple linear transformation. Thus, w and $\Gamma * F$ are solution of $-\Delta u = F$, which implies $w \equiv \Gamma * F$ by uniqueness. Then, by our earlier estimates on the Newtonian potential, we obtain

$$
\|D^2 w\|_{L^p(B_{2\delta}(x_0))} = \|D^2 w\|_{L^p(\mathbb{R}^n)} \le C\|F\|_{L^p(\mathbb{R}^n)} = C\|F\|_{L^p(B_{2\delta}(x_0))}. \tag{3.17}
$$

Estimating each term in F yields

$$
\|F\|_{L^p(B_{2\delta}(x_0))} \le \epsilon(2\delta)\|D^2 w\|_{L^p(B_{2\delta}(x_0))} + \|f\|_{L^p(B_{2\delta}(x_0))} + C\left(\|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right).
$$

Combining this estimate with the estimate (3.17) and choosing δ sufficiently small so that $C\epsilon(2\delta) < 1/2$, we have

$$
\|D^2 w\|_{L^p(B_{2\delta}(x_0))} \le \frac{1}{2}\|D^2 w\|_{L^p(B_{2\delta}(x_0))} + C\left(\|f\|_{L^p(B_{2\delta}(x_0))} + \|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right),
$$

which is equivalent to

$$
\|D^2 w\|_{L^p(B_{2\delta}(x_0))} \le C\left(\|f\|_{L^p(B_{2\delta}(x_0))} + \|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right).
$$

Hence,

$$
\|D^2 u\|_{L^p(B_{\delta}(x_0))} \le C\left(\|f\|_{L^p(B_{2\delta}(x_0))} + \|Du\|_{L^p(B_{2\delta}(x_0))} + \|u\|_{L^p(B_{2\delta}(x_0))}\right),
$$

where we used the fact that $\|D^2 u\|_{L^p(B_\delta(x_0))} = \|D^2 w\|_{L^p(B_\delta(x_0))}$ since $u \equiv w$ on $B_\delta(x_0)$.

We can easily extend this estimate from a δ-ball to any compact subset K of U via a standard covering argument. Namely, for any compact subset $K \subset U$, let $\delta < \frac{1}{2}dist(K, \partial U)$, then $K \subset U_{2\delta}$ and we can derive the desired interior estimate:

$$
\|D^2 u\|_{L^p(K)} \le C\left(\|f\|_{L^p(U)} + \|Du\|_{L^p(U)} + \|u\|_{L^p(U)}\right).
$$

Part II: Boundary Estimates.

The main ideas used in establishing the boundary estimate are relatively similar to the proof of the interior estimate. Roughly speaking, we may flatten out the boundary and treat the regularity problem as one on an upper half-space. We refer the reader to [5, 6, 11] for more details and we only sketch the main steps here. More precisely, for any point $x_0 \in \partial U$, the intersection $B_\delta(x_0) \cap \partial U$ is a $C^{2,\alpha}$ graph for $\delta > 0$ small enough. Therefore, after flattening out the boundary, we may assume that this graph is given by

$$x_n = h(x_1, x_2, \ldots, x_{n-1}) = h(x'),$$

and U lies on top of this graph locally. Now let $y = \psi(x) = (x' - x_0', x_n - h(x'))$ so that ψ is a diffeomorphism mapping a neighborhood of x_0 onto the upper ball $B_r^+(0) = \{y \in B_r(0) \,|\, y_n > 0\}$. Under this map, the elliptic equation becomes

$$\begin{cases} -\bar{a}^{ij}(y) D_{ij} u(y) + \bar{b}_i(y) D_i u(y) + \bar{c}(y) u(y) = \bar{f}(y) & \text{in } B_r^+(0), \\ u(y) = 0 & \text{on } \partial B_r^+(0). \end{cases} \tag{3.18}$$

Here the coefficients come from the original coefficients under the diffeomorphism ψ. For example, using the chain rule,

$$\bar{a}^{ij}(y) = \frac{\partial \psi^i}{\partial x^\ell}(\psi^{-1}(y)) a^{\ell k}(\psi^{-1}(y)) \frac{\partial \psi^j}{\partial x^k}(\psi^{-1}(y)).$$

We can assume $\bar{a}^{ij}(0) = \delta_{ij}$ otherwise we can apply a linear transformation to ensure this property holds. Moreover, since planes are mapped to planes under this diffeomorphism, we can assume problem (3.18) is valid even for smaller r. Applying the method of frozen coefficients with $w(y) = \varphi(2|y|/r)u(y)$ yields

$$-\Delta w(y) = F(y) \text{ in } B_r^+(0).$$

Now let $\bar{w}(y)$ and $\bar{F}(y)$, respectively, be the odd extension of $w(y)$ and $F(y)$ from $B_r^+(0)$ to $B_r(0)$. More precisely,

$$\bar{w}(y) := \begin{cases} w(y_1, y_2, \ldots, y_{n-1}, y_n) & \text{if } y_n \geq 0, \\ -w(y_1, y_2, \ldots, y_{n-1}, -y_n) & \text{if } y_n < 0. \end{cases}$$

and

$$\bar{F}(y) := \begin{cases} F(y_1, y_2, \ldots, y_{n-1}, y_n) & \text{if } y_n \geq 0, \\ -F(y_1, y_2, \ldots, y_{n-1}, -y_n) & \text{if } y_n < 0. \end{cases}$$

We can show that

$$-\Delta \bar{w}(y) = \bar{F}(y) \text{ in } B_r(0).$$

Thus, we can apply the same arguments as before to get the basic interior estimate for this problem, i.e.,

$$\begin{aligned}\|D^2u\|_{L^p(B_r(x_0))} &\le C(\|f\|_{L^p(B_{2r}(x_0))} + \|Du\|_{L^p(B_{2r}(x_0))} + \|u\|_{L^p(B_{2r}(x_0))})\\ &\le C(\|f\|_{L^p(B_{2r}(x_0)\cap U)} + \|Du\|_{L^p(B_{2r}(x_0)\cap U)} + \|u\|_{L^p(B_{2r}(x_0)\cap U)}),\end{aligned}$$

and this holds for any x_0 on ∂U and for some small radius $r > 0$. Note that the last line of the previous estimate follows from the symmetric extension of w to $\bar{w}$ from the half ball to the whole ball.

Furthermore, these balls form a covering of the boundary ∂U. By compactness of this boundary, there is a finite cover $B_{r_i}(x_i)$, $i = 1, 2, \dots k$. These balls also cover a neighborhood of ∂U including $U \backslash U_\delta$ for some suitably small $\delta > 0$. Summing the estimates over each ball in the finite cover will imply the desired boundary estimate

$$\|u\|_{W^{2,p}(U\backslash U_\delta)} \le C(\|Du\|_{L^p(U)} + \|u\|_{L^p(U)} + \|f\|_{L^p(U)}).$$

This completes the proof of the $W^{2,p}$ a priori estimates.

3.2.2 Regularity of Solutions and A Priori Estimates

Let $1 < p < \infty$. So far, we have established a priori estimates to solutions in the $W^{2,p}(U)$ norm by assuming weak solutions were already strong solutions belonging to $H_0^1(U) \cap W^{2,p}(U)$. Here we shall only assume u is a weak solution in $W_0^{1,p}(U)$. Then we actually show that u necessarily belongs to $W^{2,p}(U)$ with the help of the a priori estimates. The procedure for doing so has many points in common with our earlier derivations of the $W^{2,p}$ a priori estimates but with some subtle differences.

We say $u \in W_0^{1,p}(U)$ is a weak solution of

$$\begin{cases} Lu = f & \text{in } U,\\ u = 0 & \text{on } \partial U,\end{cases} \tag{3.19}$$

if for any $v \in W_0^{1,q}(U)$ with $1/p + 1/q = 1$,

$$\int_U \left[a^{ij}(x)D_iuD_jv + b^i(x)(D_iu)v + c(x)uv\right] dx = \int_U f(x)v\,dx.$$

Although this notion of weak solution relies on duality to define the equation in the distribution sense, the density of $C_0^\infty(U)$ in $W_0^{1,q}(U)$ ensures it is enough for the identity to hold for all test functions $v \in C_0^\infty(U)$. Our main result is the following.

Theorem 3.9. *Let $n \ge 2$ and $1 < p < \infty$ and let $U \subset \mathbb{R}^n$ be a bounded and open subset. Suppose that L is a uniformly elliptic operator whose leading coefficient $a^{ij}(x)$ is Lipschitz continuous in U, and the lower-order terms $b^i(x)$ and $c(x)$ are bounded functions in U. If $u \in W_0^{1,p}(U)$ is a weak solution of the boundary value problem* (3.19) *where $f \in L^p(U)$, then $u \in W^{2,p}(U)$.*

We shall see that the uniqueness of weak solutions of (3.19) is an important ingredient in establishing our regularity result. We only consider the case $p \geq 2$ since the uniqueness of solutions is simpler in this situation. The reason is that the uniqueness of weak solutions will allow us to improve the a priori estimates.

Lemma 3.4. *Assume that if $u \in W^{1,p}(U)$ is a weak solution of*

$$Lu = f(x) \text{ in } U, \tag{3.20}$$

then the a priori estimate

$$\|u\|_{W^{2,p}(U)} \leq C(\|u\|_{L^p(U)} + \|f\|_{L^p(U)})$$

holds. In addition, assume uniqueness holds in the sense that if $Lu = 0$, then $u \equiv 0$ in U. Then, for the unique solution u of (3.20)*, we obtain the refined a priori estimate*

$$\|u\|_{W^{2,p}(U)} \leq C\|f\|_{L^p(U)}. \tag{3.21}$$

Proof. Assume the inequality (3.21) is false. That is, there exists a sequence of functions (f_k) with $\|f\|_{L^p(U)} = 1$ and the sequence of corresponding solutions (u_k) satisfying

$$Lu_k = f_k(x) \text{ in } U,$$

such that

$$\|u_k\|_{W^{2,p}(U)} \longrightarrow \infty \text{ as } k \longrightarrow \infty.$$

We consider the normalized functions

$$v_k := u_k/\|u_k\|_{L^p(U)} \text{ and } g_k := f_k/\|u_k\|_{L^p(U)}.$$

Thus,

$$\|v_k\|_{L^p(U)} = 1 \text{ and } \|g_k\|_{L^p(U)} \longrightarrow 0 \text{ as } k \longrightarrow \infty, \tag{3.22}$$

and

$$Lv_k = g_k(x) \text{ in } U. \tag{3.23}$$

Of course, we have the a priori estimate

$$\|v_k\|_{W^{2,p}(U)} \leq C(\|v_k\|_{L^p(U)} + \|g_k\|_{L^p(U)}).$$

Combining this with (3.22) shows (v_k) is bounded in $W^{2,p}(U)$ and so the Banach-Alaoglu theorem implies there exists a subsequence, which we still label as (v_k), that converges weakly to some $v \in W^{2,p}(U)$. On the other hand, the compact Sobolev embedding implies that the same subsequence converges strongly to $v \in L^p(U)$, and hence $\|v\|_{L^p(U)} = 1$. Sending $k \longrightarrow \infty$ in (3.23) shows

$$Lv = 0 \text{ in } U.$$

By the uniqueness assumption, $v \equiv 0$, but this contradicts with $\|v\|_{L^p(U)} = 1$. This completes the proof. $\square$

Proposition 3.4. *Let $p \geq 1$ and assume $f \in L^p(B_1(0))$. Then the Dirichlet problem*

$$\begin{cases} -\Delta u = f & \text{in } B_1(0), \\ \quad u = 0 & \text{on } \partial B_1(0), \end{cases} \tag{3.24}$$

has a unique solution $u \in W^{2,p}(B_1(0))$ satisfying

$$\|u\|_{W^{2,p}(B_1(0))} \leq C\|f\|_{L^p(B_1(0))}. \tag{3.25}$$

Proof. Uniqueness follows by testing the equation against u, integrating over $B_1(0)$ then integrating by parts to get

$$\int_{B_1(0)} |Du|^2 \, dx = 0.$$

Thus, $Du \equiv 0$ and so u is constant in $B_1(0)$. The boundary condition further implies that $u \equiv 0$.

Since f is continuous, the existence of solutions follows from the integral representation,

$$u(x) = \int_{B_1(0)} G(x, y) f(y) \, dy, \; x \in B_1(0),$$

where $G(x, y)$ is the Green's function for the region $B_1(0)$. More precisely,

$$G(x, y) = \Gamma(y - x) - \phi^x(y)$$

where $\Gamma(x)$ is the fundamental solution of Laplace's equation and $\phi^x(y)$, when $n \geq 3$, is the corrector function (c.f., (1.48))

$$\phi^x(y) = \frac{1}{(n-2)\omega_n}(|x||x/|x|^2 - y|)^{2-n}.$$

It remains to show the $W^{2,p}$ estimate for this integral representation of the solution. Of course, we have already established the estimate for the first part

$$\int_{B_1(0)} \Gamma(x - y) f(y) \, dy$$

since this is just the Newtonian potential of $f(x)$, but we are missing the estimate for the part involving the corrector function. Instead, we proceed with an approximation argument. For $\delta > 0$ suitably small, consider the ball $B_{1-\delta}(0)$ and set

$$u_\delta(x) = \int_{B_1(0)} G(x, y) f_\delta(y) \, dy,$$

where

$$f_\delta(x) := \begin{cases} f(x) & \text{if } x \in B_{1-\delta}(0), \\ 0 & \text{elsewhere.} \end{cases}$$

From our earlier result on Newtonian potentials, there holds that $D^2 u_\delta$ belongs to $L^p(B_1(0))$. Thus, by Poincaré's inequality, u_δ belongs to $L^p(B_1(0))$ and hence, to $W^{2,p}(B_1(0))$ as well. From Lemma 3.4, we have the improved a priori estimate

$$\|u_\delta\|_{W^{2,p}(B_1(0))} \le C\|f_\delta\|_{L^p(B_1(0))}.$$

We may choose a sequence $\{\delta_i\} \longrightarrow 0^+$ so that the corresponding solutions $\{u_{\delta_i}\}$ is a Cauchy sequence in $W^{2,p}(B_1(0))$. This follows since

$$\|u_{\delta_i} - u_{\delta_j}\|_{W^{2,p}(B_1(0))} \le C\|f_{\delta_i} - f_{\delta_j}\|_{L^p(B_1(0))} \longrightarrow 0$$

as $i, j \longrightarrow \infty$. Then let u_0 be the limit point of this Cauchy sequence in $W^{2,p}(B_1(0))$. Then $u_0 \in W^{2,p}(B_1(0))$ is a solution of (3.24) and the improved a priori estimate (3.25) holds. This completes the proof.

□

Proof of Theorem 3.9. In view of our comments above, assume that $p \ge 2$. Consider the usual smooth cut-off function

$$\varphi(s) := \begin{cases} 1 & \text{if } s \le 1, \\ 0 & \text{if } s \ge 2. \end{cases}$$

Let $u \in W_0^{1,p}(U)$ be a weak solution of (3.19). For any x_0 in $U_{2\delta} := \{x \in U \mid dist(x, \partial U) \ge 2\delta\}$, let

$$\eta(x) = \varphi\Big(\frac{|x - x_0|}{\delta}\Big) \text{ and } w(x) = \eta(x)u(x).$$

Thus, w is supported in $B_{2\delta}(x_0)$. By our definition of a weak solution in $W_0^{1,p}(U)$, it is easily verified that for any $v \in C_0^\infty(B_{2\delta}(x_0))$,

$$\int_{B_{2\delta}(x_0)} a^{ij}(x_0) D_i w D_j v \, dx = \int_{B_{2\delta}(x_0)} [a^{ij}(x_0) - a^{ij}(x)] D_i w D_j v + F(x) v \, dx,$$

where

$$F(x) = f(x) - D_j(a^{ij}(x)(D_i\eta)u) - b^i(x) D_i u - c(x)u.$$

Namely, w is a weak solution of

$$\begin{cases} -a^{ij}(x_0) D_{ij} w = -D_j([a^{ij}(x_0) - a^{ij}(x)] D_i w) + F(x) & \text{in } B_{2\delta}(x_0), \\ \qquad w = 0 & \text{on } \partial B_{2\delta}(x_0). \end{cases} \tag{3.26}$$

As before, we may assume $a^{ij}(x_0) = \delta_{ij}$ and we may rewrite (3.26) as

$$\begin{aligned} -\Delta w = & \; - D_j([a^{ij}(x_0) - a^{ij}(x)] D_i w) + F(x) \\ = & \; - [a^{ij}(x_0) - a^{ij}(x)] D_{ij} w + \tilde{F}(x) \text{ in } B_{2\delta}(x_0), \end{aligned} \tag{3.27}$$

where

$$\tilde{F}(x) = D_j[a^{ij}(x)]D_i w + F(x).$$

For any $v \in W^{2,p}(B_{2\delta}(x_0))$, clearly

$$[a^{ij}(x_0) - a^{ij}(x)]D_{ij}v \in L^p(B_{2\delta}(x_0)).$$

In addition, it is easy to verify that $\tilde{F}$ belongs to $L^p(B_{2\delta}(x_0))$. In view of Proposition 3.4, the Laplacian Δ is an invertible linear operator, and so we may consider the equation

$$v = Kv + (-\Delta)^{-1}\tilde{F} \quad \text{in } W^{2,p}, \tag{3.28}$$

where

$$Kv(x) := \Delta^{-1}([a^{ij}(x_0) - a^{ij}(x)]D_{ij}v).$$

From the Lipschitz continuity of $a^{ij}(x)$, K is a contraction mapping from $W^{2,p}(B_{2\delta}(x_0))$ to itself provided that $\delta > 0$ is sufficiently small. Thus, there exists a unique solution $v \in W^{2,p}(B_{2\delta}(x_0))$ to equation (3.28). By the uniqueness of solutions of (3.27), which follows from arguments similar to those in the proof of Proposition 3.4, we have that $w \equiv v$ in $W^{2,p}(B_{2\delta}(x_0))$. Therefore, the regularity of u holds locally in a neighborhood of $x_0 \in U$. Since x_0 was chosen arbitrarily and since U is bounded, a standard covering argument yields the regularity of u up to the entire domain. That is, u belongs to $W^{2,p}(U)$. □

Remark 3.5. *In summary, a priori regularity estimates imply the actual regularity of weak solutions. From this point on, we study the regularity of solutions in various settings and function spaces, however, we only establish the a priori estimates. It should be understood that the actual regularity of the solutions will follow from the a priori estimates using similar ideas in this section.*

3.3 Bootstraping: Two Basic Examples

We show how to combine the previous $W^{2,p}$ a priori estimates with the Hölder estimates of Sections 1.4.2 and 1.4.3 (or more generally the Schauder estimates of Section 3.5 below) to get the smoothness of weak solutions to a simple linear PDE and a related semilinear problem. The goal here is to introduce and provide simple examples of bootstrap methods.

Let $n \geq 3$ and suppose $U \subset \mathbb{R}^n$ is a bounded open subset with C^1 boundary. Consider the linear problem

$$\begin{cases} -\Delta u = c(x)u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3.29}$$

and the semilinear problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{3.30}$$

We shall prove that if $u \in H_0^1(U)$ is a weak solution of either problem, then it is actually smooth and therefore a classical solution. The idea is to treat each PDE as a linear equation with an integrable coefficient, then we apply the Sobolev embedding recursively to boost the smoothness of u and verify it is Hölder continuous. The Schauder estimates will then show u is of class $C^{2,\alpha}$. Similarly, applying the Schauder estimates successively will further imply that the solution is in fact smooth.

Remark 3.6. *This idea of starting with a solution residing in a lower regularity space and iterating the a priori estimates to show it actually belongs to a higher regularity space is an example of a bootstrap procedure. We shall revisit bootstrap arguments again in the subsequent sections.*

Theorem 3.10. *Suppose that $u \in H_0^1(U)$ is a weak solution of problem* (3.29) *and $c(x)$ belongs to $L^{\frac{n}{2}}(U)$. Then u is smooth, i.e., $u \in C^\infty$.*

Proof. By the Sobolev inequality, u belongs to $L^{\frac{2n}{n-2}}(U)$. Thus, Hölder's inequality ensures the source term $c(x)u$ belongs to $L^{\frac{2n}{n+2}}(U)$, since

$$\|cu\|_{L^{\frac{2n}{n+2}}(U)} \leq \|c\|_{L^{\frac{n}{2}}(U)}\|u\|_{L^{\frac{2n}{n-2}}(U)}.$$

Then the L^p regularity theory implies $u \in W^{2,s_0}(U)$ where $s_0 = 2n/(n+2)$. Again, the Sobolev embedding $W^{2,s}(U) \hookrightarrow L^{\frac{ns}{n-2s}}(U)$ implies that u belongs to $L^{s_1}(U)$ and thus belongs to $W^{2,s_1}(U)$, where $s_1 = ns_0/(n-2s_0)$. If $s_1 > n$, Sobolev embedding, particularly Morrey's inequality, implies that u belongs to $C^\alpha(U)$ where $\alpha = 1 - n/s_1 \in (0,1)$; otherwise, if $s_1 \leq n$, we can invoke the L^p theory and the Sobolev embedding once again to deduce that $u \in W^{2,s_1}(U) \hookrightarrow L^{s_2}(U)$, where $s_2 = ns_1/(n-2s_1) = ns_0/(n-4s_0)$. Therefore, if $s_2 > n$, we get that u belongs to $C^\alpha(U)$ for some $\alpha \in (0,1)$ and we are done. Otherwise, we may repeat this argument successively to find a suitably large j in which $s_j > n$ and u belongs to $W^{2,s_j}(U)$. Hence, Sobolev embedding ensures $u \in C^\alpha(U)$ for some $\alpha \in (0,1)$. By applying the Schauder estimates repeatedly, we deduce that u is smooth. □

A consequence of this result is the smoothness of weak solutions to problem (3.30).

Corollary 3.1. *Suppose $1 < p < (n+2)/(n-2)$. If $u \in H_0^1(U)$ is a weak solution of problem* (3.30)*, then u is smooth.*

Proof. Set $c(x) = |u|^{p-1}$. Since u belongs to $H_0^1(U)$, the Sobolev inequality implies that $u \in L^s(U)$ for $1 \leq s \leq 2n/(n-2)$. From this, it is easy to check that $c(x)$ belongs to $L^{\frac{n}{2}}(U)$. Hence, the previous theorem applies to show u is smooth. □

3.4 Regularity in the Sobolev Spaces H^k

In this section, we show the regularity of weak solutions to uniformly elliptic equations in $H^2(U)$ or $W^{2,2}(U)$. Under the appropriate conditions, we shall establish both interior and boundary a priori estimates for the weak solutions to conclude that they are indeed strong solutions. Then, we iterate these estimates under the right conditions to conclude that the weak solutions belong to higher order Sobolev spaces. In fact, we show weak solutions are actually classical solutions if the data of the elliptic problem are smooth. We assume throughout the section that $U \subset \mathbb{R}^n$ is a bounded, open set and we take $u \in H_0^1(U)$ to be a weak solution of

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

where as always L is uniformly elliptic and is in divergence form, i.e.,

$$Lu = -\sum_{i,j=1}^{n} D_j \left(a^{ij}(x) D_i u\right) + \sum_{j=1}^{n} b^i(x) D_i u + c(x) u.$$

Of course, the regularity of the coefficients a^{ij}, b^i and c and the source term f must be specified for each regularity result.

3.4.1 Interior regularity

Theorem 3.11 (Interior H^2-regularity)**.** *Assume*

$$a^{ij} \in C^1(U),\, b^i,\, c \in L^\infty(U) \quad \textit{for } i,j = 1,2,\ldots,n, \tag{3.31}$$

and $f \in L^2(U)$. Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \;\; \textit{in } U.$$

Then u belongs to $H^2_{loc}(U)$ and thus is a strong solution of this elliptic PDE, and for each open subset $V \subset\subset U$ there holds the estimate

$$\|u\|_{H^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}), \tag{3.32}$$

where the positive constant C depends only on V, U and the coefficients of the operator L.

Remark 3.7. *Note that this theorem is not assuming u satisfies the Dirichlet boundary condition on ∂U. Also, recall that u is said to be a strong solution of the elliptic PDE if it is twice weakly differentiable and satisfies the equation $Lu = f$, for a.e. x in U. Indeed, this follows simply from the fact that u belongs to $H^2_{loc}(U)$. More precisely, the definition of a weak solution and integration by parts indicates that*

$$(Lu, v) = B[u, v] = (f, v)$$

for all $v \in C_c^\infty(U)$. Thus, from Corollary A.2, this shows that $Lu - f = 0$ a.e. or that $Lu = f$ for a.e. $x \in U$.

Proof of Theorem 3.11. Fix $V \subset\subset U$, choose an open W such that $V \subset\subset W \subset\subset U$, and select a smooth cut-off function ζ such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in V and $\zeta \equiv 0$ in W^C.

Step 1: Since $u \in H^1(U)$ is a weak solution of $Lu = f$ in U, there holds

$$\sum_{i,j=1}^{n} \int_U a^{ij}(x) D_i u D_j v \, dx = \int_U F v \, dx \;\text{ for every }\; v \in H_0^1(U), \tag{3.33}$$

where

$$F := f - \sum_{i=1}^{n} b^i(x) D_i u - c(x) u.$$

Step 2: Let $|h| > 0$ be small, choose $k \in \{1, 2, \ldots, n\}$ and substitute

$$v = -D_k^{-h}(\zeta^2 D_k^h u)$$

into (3.33) where $D_k^h u$ is the difference quotient

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h} \;\; (h \in \mathbb{R}\backslash\{0\}).$$

For this particular test function v, we denote the resulting left-hand side (respectively, right-hand side) of (3.33) by A (respectively, B). After some tedious calculations and denoting $v^h(x) := v(x + he_k)$, we calculate

$$\begin{aligned} A = &\sum_{i,j=1}^{n} \int_U a^{ij,h}(x) D_k^h D_i u D_k^h D_j u \zeta^2 \, dx + \sum_{i,j=1}^{n} \int_U [a^{ij,h}(x) D_k^h D_i u D_k^h u (2\zeta) D_j \zeta \\ &+ (D_k^h a^{ij}(x)) D_i u D_k^h D_j u \zeta^2 + (D_k^h a^{ij}(x)) D_i u D_k^h u (2\zeta) D_j \zeta] \, dx \\ =:& \, A_1 + A_2. \end{aligned}$$

Indeed, the uniform ellipticity condition implies

$$A_1 \ge \theta \int_U \zeta^2 |D_k^h D u|^2 \, dx.$$

In addition, from (3.31) we get

$$|A_2| \le C \int_U \zeta |D_k^h D u| |D_k^h u| + \zeta |D_k^h D u| |D u| + \zeta |D_k^h u| |D u| \, dx,$$

for some constant $C > 0$. Thus, Cauchy's inequality with ϵ (see Theorem A.1) implies the estimate

$$|A_2| \le \epsilon \int_U \zeta^2 |D_k^h D u|^2 \, dx + \frac{C}{\epsilon} \int_W |D_k^h u|^2 + |D u|^2 \, dx.$$

Choosing $\epsilon = \theta/2$ and using the fact that

$$\int_W |D_k^h u|^2 \, dx \leq C \int_U |Du|^2 \, dx,$$

we arrive at

$$|A_2| \leq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \, dx + C \int_U |Du|^2 \, dx.$$

This estimate and the estimate of A_1 imply

$$A \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \, dx - C \int_U |Du|^2 \, dx. \tag{3.34}$$

Recalling the definition of F and our particular choice of the test function v, we get

$$\begin{aligned} |B| &\leq C \int_U (|f| + |Du| + |u|)|v| \, dx \\ &\leq \epsilon \int_U \zeta^2 |D_k^h Du|^2 \, dx + \frac{C}{\epsilon} \int_U f^2 + u^2 + |Du|^2 \, dx \end{aligned}$$

where we used Cauchy's inequality with ϵ (Theorem A.1) and the fact that

$$\int_U |v|^2 \, dx \leq C \int_U |Du|^2 + \zeta^2 |D_k^h Du|^2 \, dx.$$

Choosing $\epsilon = \theta/4$, we arrive at

$$|B| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 \, dx + C \int_U f^2 + u^2 + |Du|^2 \, dx \leq C(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2). \tag{3.35}$$

Recalling that $A = B$ and inserting the estimates (3.34) and (3.35), we deduce that

$$\int_V |D_k^h Du|^2 \, dx \leq \int_U \zeta^2 |D_k^h Du|^2 \, dx \leq C \int_U f^2 + u^2 + |Du|^2 \, dx$$

for $k = 1, 2, \ldots, n$, and all sufficiently small $|h| \neq 0$. This implies that $Du \in H^1_{loc}(U; \mathbb{R}^n)$. Hence, we have that $u \in H^2_{loc}(U)$ with the estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{3.36}$$

Step 3: Notice that we are not quite done; namely, it remains to replace the H^1 norm of u instead with its L^2 norm in the estimate (3.36).

Indeed, since $V \subset\subset W \subset\subset U$, the procedure above can be used to establish the interior estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}) \tag{3.37}$$

for an appropriate positive constant C depending on V, W, etc. Choosing a new smooth cut-off function $0 \leq \zeta \leq 1$ with $\zeta \equiv 1$ in W, $supp(\zeta) \subset U$ and setting $v = \zeta^2$ in identity (3.33), elementary calculations will lead to the estimate

$$\int_U \zeta^2 |Du|^2 \, dx \leq C \int_U f^2 + u^2 \, dx.$$

Hence,

$$\|u\|_{H^1(W)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}),$$

and inserting this into (3.37) completes the proof of the theorem. □

3.4.2 Higher interior regularity

By assuming stronger smoothness of the coefficients in the elliptic equation, we may iterate the previous interior regularity theorem to get the higher regularity of weak solutions. Namely, there holds the following.

Theorem 3.12 (Higher interior regularity)**.** *Let m be a non-negative integer, and assume*

$$a^{ij},\, b^i,\, c \in C^{m+1}(U) \quad \text{for } i, j = 1, 2, \ldots, n,$$

and

$$f \in H^m(U).$$

Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \;\text{ in } U.$$

Then

$$u \text{ belongs to } H^{m+2}_{loc}(U), \tag{3.38}$$

and for each open subset $V \subset\subset U$ there holds the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)}), \tag{3.39}$$

where the positive constant C depends only on m, V, U and the coefficients of the elliptic operator L.

Proof. We proceed by induction. Clearly, the case $m = 0$ holds by Theorem 3.11.
Step 1: Assume that assertions (3.38) and (3.39) hold for an arbitrary integer $m \geq 2$ and all open sets U, coefficients a^{ij}, b^i, c, etc. Now suppose

$$a^{ij},\, b^i,\, c \in C^{m+2}(U), \tag{3.40}$$

and

$$f \in H^{m+1}(U), \tag{3.41}$$

and $u \in H^1(U)$ is a weak solution of $Lu = f$ in U.

So by the induction hypothesis, there holds $u \in H^{m+2}_{loc}(U)$ with the interior estimate

$$\|u\|_{H^{m+2}_{loc}(W)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}) \tag{3.42}$$

for each $W \subset\subset U$ and an appropriate positive constant C, depending only on W, the coefficients of L, etc. Now fix $V \subset\subset W \subset\subset U$.

Step 2: Now let α be any multi-index with $|\alpha| = m + 1$, and choose any test function $v_1 \in C_c^\infty(W)$. Inserting $v := (-1)^{|\alpha|} D^\alpha v_1$ into the weak solution definition $B[u, v] = (f, v)_{L^2(U)}$, elementary calculations will lead to the identity

$$B[u_1, v_1] = (f_1, v_1)_{L^2(U)} \tag{3.43}$$

where

$$u_1 := D^\alpha u \in H^1(W) \tag{3.44}$$

and

$$\begin{aligned} f_1 := D^\alpha f - \sum_{\beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \Big[- \sum_{i,j=1}^n D_j \Big(D^{\alpha-\beta} a^{ij}(x) D^\beta D_i u \Big) \\ + \sum_{i=1}^n D^{\alpha-\beta} b^i(x) D^\beta D_i u + D^{\alpha-\beta} c(x) D^\beta u \Big]. \end{aligned} \tag{3.45}$$

Since (3.43) holds for each $v_1 \in C_c^\infty(W)$, we see that u_1 is a weak solution of $Lu = f_1$ in W. So in view of (3.40)–(3.42) and (3.44), we have $f_1 \in L^2(U)$ with

$$\|f_1\|_{L^2(W)} \leq C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}).$$

Step 3: From Theorem 3.11, we conclude that u_1 belongs to $H^2(V)$ with the estimate

$$\|u_1\|_{H^2(V)} \leq C(\|f_1\|_{L^2(W)} + \|u_1\|_{L^2(W)}) \leq C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}).$$

Since this estimate holds for each multi-index α with $|\alpha| = m+1$ and $u_1 = D^\alpha u$, we deduce that $u \in H^{m+3}(V)$ and

$$\|u\|_{H^{m+3}(V)} \leq C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}).$$

This completes the induction step for the case $m + 1$, and this finishes the proof of the theorem. $\square$

In fact, provided that the data of the problem are smooth, we can apply Theorem 3.12 successively to deduce that the weak solutions are actually smooth.

Theorem 3.13 (Infinite differentiability in the interior). *Assume*

$$a^{ij},\ b^i,\ c \in C^\infty(U) \quad for\ i,j = 1,2,\ldots,n$$

and

$$f \in C^\infty(U).$$

Suppose further that $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f \ \ in\ U.$$

Then u belongs to $C^\infty(U)$.

Proof. According to Theorem 3.12, u belongs to $H^m_{loc}(U)$ for each integer $m = 1, 2, \ldots$. So by the general Sobolev inqualities (see Theorem A.17), u belongs to $C^k(U)$ for $k = 1, 2, \ldots$. This completes the proof. □

3.4.3 Global regularity

Next, we extend the earlier interior regularity estimates up to the boundary, but not surprisingly, additional smoothness up to the boundary ∂U on the data of the problem are needed.

Theorem 3.14 (Boundary H^2-regularity). *Assume*

$$a^{ij} \in C^1(\bar{U}),\ b^i,\ c \in L^\infty(U) \quad for\ i,j = 1,2,\ldots,n, \tag{3.46}$$

$f \in L^2(U)$ and the boundary ∂U is C^2. Suppose that $u \in H^1_0(U)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & in\ U, \\ \ \ u = 0 & on\ \partial U. \end{cases} \tag{3.47}$$

Then $u \in H^2(U)$, and there holds the estimate

$$\|u\|_{H^2(U)} \le C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}), \tag{3.48}$$

where the positive constant C depends only on U and the coefficients of L.

Remark 3.8. *Note that we are now prescribing a Dirichlet boundary condition on the solution of* (3.47)*. This boundary condition, of course, should be understood in the trace sense. In addition, if u is the unique weak solution of the Dirichlet problem, then estimate* (3.48) *simplifies to*

$$\|u\|_{H^2(U)} \le C\|f\|_{L^2(U)},$$

since Theorem 2.9 implies that $\|u\|_{L^2(U)} \le \tilde{C}\|f\|_{L^2(U)}$ where $\tilde{C}$ depends only on U and the coefficients of L.

Proof of Theorem 3.14. We first prove the theorem for the special case when U is the half-ball

$$U = B_1(0) \cap \mathbb{R}^n_+.$$

Step 1: Set $V = B_{1/2}(0) \cap \mathbb{R}^n_+$ and select a smooth cut-off function ζ for which $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ in $B_{1/2}(0)$, and $\zeta \equiv 0$ in $B_1(0)^C$. In particular, $\zeta \equiv 1$ in V and vanishes near the curved part of ∂U. Since u is a weak solution of (3.47), we have that

$$B[u, v] = (f, v) \text{ for all } v \in H^1_0(U),$$

and so

$$\sum_{i,j=1}^{n} \int_U a^{ij}(x) D_i u D_j v \, dx = \int_U F v \, dx, \tag{3.49}$$

where

$$F := f - \sum_{i=1}^{n} b^i(x) D_i u - c(x) u.$$

Step 2: Now let $h > 0$ be small, choose $k \in \{1, 2, \ldots, n-1\}$ and write

$$v := -D_k^{-h}(\zeta^2 D_k^h u).$$

Note that

$$\begin{aligned} v(x) = & -\frac{1}{h} D_k^{-h}(\zeta^2(x)[u(x + he_k) - u(x)]) \\ = & \frac{1}{h^2}\Big(\zeta^2(x - he_k)[u(x) - u(x - he_k)] - \zeta^2(x)[u(x + he_k) - u(x)]\Big) \ \ (x \in U). \end{aligned}$$

Then, since $u = 0$ along $\{x_n = 0\}$ in the trace sense and $\zeta = 0$ near the curved portion of ∂U, we get that $v \in H^1_0(U)$. Then, substituting this particular choice of v into (3.49), we may write the resulting expression as $A = B$ where

$$A := \sum_{i,j=1}^{n} \int_U a^{ij}(x) D_i u D_j v \, dx \tag{3.50}$$

and

$$B := \int_U F v \, dx. \tag{3.51}$$

Step 3: We estimate the terms A and B, but the steps are similar to the steps found in the proof of Theorem 3.11 so we omit the details. Namely, there holds

$$A \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h D u|^2 \, dx - C \int_U |Du|^2 \, dx \tag{3.52}$$

and

$$|B| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h D u|^2 \, dx + C \int_U f^2 + u^2 + |Du|^2 \, dx, \tag{3.53}$$

for an appropriate positive constant C. Inserting estimates (3.52) and (3.53) into the expression $A = B$, we deduce

$$\int_V |D_k^h Du|^2 \, dx \le C \int_U f^2 + u^2 |Du|^2 \, dx$$

for $k = 1, 2, \dots, n-1$. Thus, this implies that

$$D_k u \in H^1(V) \text{ for } k = 1, 2, \dots, n-1$$

with the estimate

$$\sum_{k,\ell=1, k+\ell<2n}^{n} \|D_{\ell k} u\|_{L^2(V)} \le C(\|u\|_{H^1(U)} + \|f\|_{L^2(U)}). \tag{3.54}$$

Step 4: Notice that estimate (3.54) is missing the last term $\|D_{nn}u\|_{L^2(U)}$. We now estimate this term.

In view of Theorem 3.11 and the definition of the elliptic operator L, u is a strong solution of $Lu = f$ in V. That is,

$$-\sum_{i,j=1}^{n} a^{ij}(x) D_i u D_j u + \sum_{i=1}^{n} \tilde{b}^i(x) D_i u + c(x) u = f \tag{3.55}$$

where $\tilde{b}^i(x) := b^i(x) - \sum_{j=1}^n D_j a^{ij}(x)$ for $i = 1, 2, \dots, n$. From this we can solve for the last term $D_{nn}u$, i.e.,

$$a^{nn}(x) D_{nn} u = -\sum_{i,j=1,\, i+j<2n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} \tilde{b}^i(x) D_i u + c(x) - f. \tag{3.56}$$

From the uniform ellipticity condition, $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \ge \theta|\xi|^2$ for all $x \in U$, $\xi \in \mathbb{R}^n$. Thus, if we take $\xi = e_n = (0, 0, \dots, 0, 1)$ in the last estimate, we get

$$a^{nn}(x) \ge \theta > 0 \text{ in } U. \tag{3.57}$$

Hence, combining this and the assumptions (3.46) with identity (3.56) gives us

$$|D_{nn}u| \le C\Big(\sum_{i,j=1,\, i+j<2n}^{n} |D_{ij}u| + |Du| + |u| + |f| \Big) \text{ in } U. \tag{3.58}$$

Therefore, applying estimate (3.54) to this, we arrive at the estimate

$$\|u\|_{H^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}) \tag{3.59}$$

for some appropriate positive constant C.

Step 5: We drop the assumption that U is a half-ball. In general, we may choose any point $x^0 \in \partial U$ and since ∂U is C^2, we may assume, upon relabelling and reorienting the axes if necessary, that

$$U \cap B_r(x^0) = \{x \in B_r(x^0) \,|\, x_n > \gamma(x_1, x_2, \ldots, x_{n-1})\}$$

for some $r > 0$ and some C^2 function $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$. As indicated at the beginning of this chapter, we can change variables and write

$$y = \Phi(x) \text{ and } x = \Psi(y).$$

Step 6: Choose $s > 0$ so small that the half-ball $U_1 = B_s(0) \cap \{y_n > 0\}$ lies in $\Phi(U \cap B_r(x^0))$. Set

$$V_1 = B_{s/2}(0) \cap \{y_n > 0\} \tag{3.60}$$

and define

$$u_1(y) := u(\Psi(y)) \text{ for } y \in U_1.$$

Then it turns out that

$$(i)\ u_1 \in H^1(U_1), \qquad (ii)\ u_1 = 0 \text{ on } \partial U_1 \cap \{y_n = 0\} \tag{3.61}$$

where property (ii) should be understood in the trace sense. Then, after some elementary calculations, we can deduce that this u_1 is a weak solution of the PDE

$$L_1 u = f_1 \text{ in } U_1$$

where

$$f_1(y) = f(\Psi(y))$$

and

$$L_1 u = -\sum_{k,\ell=1}^{n} D_\ell(a_1^{k\ell} D_k u_1) + \sum_{k=1}^{n} b_1^k(x) D_k u + c_1(c) u$$

with

$$\begin{aligned}
a_1^{k\ell}(y) &= \sum_{r,s=1}^{n} a^{rs}(\Psi(y)) \Phi^k_{x_r}(\Psi(y)) \Phi^\ell_{x_s}(\Psi(y)) \ \ (k, \ell = 1, 2, \ldots, n), \\
b_1^k(y) &= \sum_{r=1}^{n} b^r(\Psi(y)) \Phi^k x_r(\Psi(y)) \ \ (k = 1, 2, \ldots, n), \\
c_1(y) &= c(\Psi(y)).
\end{aligned}$$

Then, it turns out that L_1 is a uniformly elliptic operator and the matrix coefficient $a_1^{k\ell}(x)$ is C^1 since Φ and Ψ are C^2 maps.

Step 7: Applying our results from Steps 1–4 to the elliptic problem $L_1 u = f_1$ in U_1 and recalling (3.60), we deduce that $u_1 \in H^2(V_1)$ with the estimate

$$\|u_1\|_{H^2(V_1)} \leq C(\|u_1\|_{L^2(U_1)} + \|f_1\|_{L^2(U_1)}),$$

and so

$$\|u\|_{H^2(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}) \tag{3.62}$$

for $V := \Psi(V_1)$.

Step 8: Finally, since ∂U is compact, we can cover it with finitely many sets $V_1, V_2, \ldots, V_N$ as above in which the estimate (3.62) holds in each V_i. Summing up these estimates over all V_i and combining the resulting estimate with the interior regularity estimate shows that $u \in H^2(U)$ with

$$\|u\|_{H^2(U)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}).$$

This completes the proof of the theorem. □

3.4.4 Higher global regularity

Theorem 3.15 (Higher boundary regularity)**.** *Let m be a non-negative integer, and assume*

$$a^{ij}, b^i, c \in C^{m+1}(\bar{U}) \quad \text{for } i, j = 1, 2, \ldots, n, \tag{3.63}$$

$$f \in H^m(U) \tag{3.64}$$

and the boundary ∂U is C^{m+2}. Suppose that $u \in H_0^1(U)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Then $u \in H^{m+2}(U)$, and there holds the estimate

$$\|u\|_{H^{m+2}(U)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)}), \tag{3.65}$$

where the positive constant C depends only on m, U and the coefficients of the elliptic operator L.

Proof. We only prove the boundary estimate for the special case when the domain is the half-ball $U = B_s(0) \cap \mathbb{R}^n_+$ for some $s > 0$. Proving it for a general domain U involves similar ideas as in the preceding theorem by straightening out the boundary and applying a standard covering argument.

Fix $t \in (0, s)$ and set $V = B_t(0) \cap \mathbb{R}^n_+$.

Step 1: We proceed by induction on the non-negative integer m with the goal of showing that (3.63) and (3.64), whenever $u = 0$ along $\{x_n = 0\}$ in the trace sense, imply $u \in H^{m+2}(V)$ with the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}),$$

for some positive constant C depending only on U, V and the coefficients of the operator L. Of course, the case $m = 0$ is a direct consequence of the preceding theorem.

Suppose then that

$$(i)\ a^{ij},\, b^i,\, c \in C^{m+2}(\bar{U}), \qquad (ii)\ f \in H^{m+1}(U), \tag{3.66}$$

u is a weak solution of

$$Lu = f \ \text{ in } \ U,$$

and u vanishes along $\{x_n = 0\}$ in the trace sense. Fix any $0 < t < r < s$ and write $W = B_r(0) \cap \mathbb{R}^n_+$. By the induction assumption, we have $u \in H^{m+2}(W)$ with

$$\|u\|_{H^{m+2}(W)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^m(U)}). \tag{3.67}$$

Furthermore, according to the interior regularity result of Theorem 3.12, $u \in H^{m+3}_{loc}(U)$.

Step 2: Let α be any multi-index with $|\alpha| = m+1$ and $\alpha_n = 0$. Then set $u_1 := D^\alpha u$, which belongs to $H^1(U)$ and vanishes along the plane $\{x_n = 0\}$ in the trace sense. Furthermore, as in the proof of Theorem 3.12, u_1 is a weak solution of

$$L_1 u = f_1,$$

where

$$f_1 := D^\alpha f - \sum_{\beta \leq \alpha,\, \beta \neq \alpha} \binom{\alpha}{\beta} \Big[\sum_{i,j=1}^n -\Big(D^{\alpha-\beta} a^{ij}(x) D^\beta D_i u \Big) + \sum_{i=1}^n D^{\alpha-\beta} b^i(x) D^\beta D_i u + D^{\alpha-\beta} c(x) D^\beta u \Big].$$

So in view of (3.63), (3.64), (3.66)(ii) and (3.67), we see that $f_1 \in L^2(W)$ with

$$\|f_1\|_{L^2(W)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)}).$$

From our proof of Theorem 3.14, we can deduce that $u_1 \in H^2(V)$ with

$$\|u_1\|_{H^2(V)} \leq C(\|u_1\|_{L^2(W)} + \|f_1\|_{L^2(W)}) \leq C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)}).$$

Noting that $u_1 = D^\alpha u$, this shows that

$$\|D^\beta u\|_{L^2(V)} \leq C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)})$$

for any multi-index β with $|\beta| = m+3$ and $\beta_n = 0, 1$, or 2.

Step 3: We only need to remove the previous restriction on β_n, and we do so by induction. Namely, assume that

$$\|D^\beta u\|_{L^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)})$$

for any multi-index β with $|\beta| = m+3$ and $\beta_n = 0, 1, 2, \ldots, j$ for some $j \in \{2, 3, \ldots, m+2\}$. Assume then $|\beta| = m+3$, $\beta_n = j+1$. Let us write $\beta = \gamma + \delta$ for $\delta = (0, \ldots, 0, 2)$ and $|\gamma| = m+1$. Since, $u \in H^{m+3}_{loc}(U)$ and $Lu = f$ in U, we have $D^\gamma Lu = D^\gamma f$ a.e. in U. Now, $D^\gamma Lu = a^{nn}(x) D^\beta u + T$ where T is a sum of terms involving at most j derivatives of u with respect to x_n and at most $m+3$ derivatives with respect to all the other variables. Since $a^{nn}(x) \ge \theta > 0$ in U, the initial induction hypothesis imply that

$$\|D^\beta u\|_{L^2(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)})$$

provided that $|\beta| = m+3$ and $\beta_n = j+1$. So by induction, we have

$$\|u\|_{H^{m+3}(V)} \le C(\|u\|_{L^2(U)} + \|f\|_{H^{m+1}(U)}).$$

This completes the proof. $\square$

We have a global smoothness property of weak solutions to the Dirichlet problem provided the data are globally smooth.

Theorem 3.16 (Infinite differentiability up to the boundary)**.** *Assume*

$$a^{ij}, b^i, c \in C^\infty(\bar{U}) \quad \textit{for } i, j = 1, 2, \ldots, n,$$

$f \in C^\infty(\bar{U})$ and the boundary ∂U is C^∞. Suppose that $u \in H^1_0(U)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Then $u \in C^\infty(\bar{U})$.

Proof. According to Theorem 3.15, we have $u \in H^m(U)$ for each integer $m = 1, 2, \ldots$. Thus, Theorem A.17 implies that u belongs to $C^k(\bar{U})$ for each $k = 1, 2, \ldots$. This completes the proof of the theorem. $\square$

3.5 The Schauder Estimates and $C^{2,\alpha}$ Regularity

This section briefly recalls results from the Schauder theory for classical solutions. The proofs of the interior and global estimates can be found in [6].

Let $U \subseteq \mathbb{R}^n$, $x_0 \in U$ and $\alpha \in (0,1]$. We denote by $C^k(\bar{U}) = C^{k,0}(\bar{U})$ the Banach space of functions f which are k-times continuously differentiable on $\bar{U}$ equipped with the norm

$$\|f\|_{k;U} := \sum_{j=0}^{k} [f]_{j;U}, \tag{3.68}$$

where $[f]_{j;U} := \sup_U |D^j f(x)|$.

For Hölder continuity, we introduce the corresponding class of spaces often called Hölder spaces. We say a function f is **Hölder continuous with exponent** α **at** x_0 if the quantity

$$[f]_{\alpha,x_0} := \sup_U \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}$$

is finite. Furthermore, if $\alpha = 1$, then f is said to be Lipschitz continuous at x_0. We say f is **Hölder continuous with exponent** α **in** U if

$$[f]_{\alpha;U} := \sup_{x,y\in U, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite. For $\alpha \in (0,1]$, we introduce the additional semi-norms

$$\begin{aligned}
[f]_{0,0;U} = [f]_{0;U} &:= \sup_U |f(x)|, \\
[f]_{0,\alpha;U} = [f]_{\alpha;U} &:= \sup_{x_0\in U} [f]_{\alpha,x_0}, \\
[f]_{k,0;U} = [f]_{k;U} &:= \sum_{|\beta|=k} [D^\beta f]_{0;U}, \\
[f]_{k,\alpha;U} &:= \sum_{|\beta|=k} [D^\beta f]_{\alpha;U}.
\end{aligned}$$

Definition 3.5. *We denote by $C^{k,\alpha}(\bar{U})$ $(0 < \alpha \leq 1)$ the space consisting of functions $f \in C^k(\bar{U})$ satisfying $[f]_{k,\alpha;U} < \infty$. This space is indeed a Banach space equipped with the norm*

$$\|f\|_{k,\alpha;U} := \|f\|_{k;U} + [f]_{k,\alpha;U}. \tag{3.69}$$

Let U be an bounded open domain and consider the general second-order linear elliptic equation

$$-a^{ij}(x)D_{ij}u + b^i(x)D_i u + c(x)u = f \ \text{ in } \ U. \tag{3.70}$$

As usual, we assume there exist $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \ \text{ for all } \ x \in U,\ \xi \in \mathbb{R}^n,$$

$a^{ij}, b^i, c \in C^\alpha(\bar{U})$ $(0 < \alpha < 1)$ and

$$\frac{1}{\lambda}\Big\{ \sum_{ij} \|a^{ij}\|_{\alpha;U} + \sum_i \|b^i\|_{\alpha;U} + \|c\|_{\alpha;U} \Big\} \leq \Lambda_\alpha.$$

Theorem 3.17 (Interior Schauder estimates). *For $\alpha \in (0,1)$, let $u \in C^{2,\alpha}(U)$ be a solution of (3.70). Then for $U' \subset\subset U$, we have*

$$\|u\|_{2,\alpha;U'} \leq C\left(\frac{1}{\lambda}\|f\|_{\alpha;U} + \|u\|_{0;U}\right),$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and $dist(U', \partial U)$.

Following similar ideas used in obtaining the interior estimates, we can establish corresponding boundary Schauder estimates.

Theorem 3.18 (Global Schauder estimates). *Consider the same assumptions from the previous theorem and further assume $\partial U \in C^{2,\alpha}$. Suppose that $u \in C^{2,\alpha}(\bar{U})$ is a solution of* (3.70) *satisfying the boundary condition $u = g$ on ∂U where $g \in C^{2,\alpha}(\bar{U})$. Then*

$$\|u\|_{2,\alpha;U} \leq C\left(\frac{1}{\lambda}\|f\|_{\alpha;U} + \|g\|_{2,\alpha;U} + \|u\|_{0;U}\right),$$

where C depends only on $n, \alpha, \Lambda/\lambda, \Lambda_\alpha$ and U. Moreover, if u satisfies the maximum principle, then the last term on the right-hand side of the global estimate can be dropped.

3.6 Hölder Continuity for Weak Solutions: A Perturbation Approach

In this section, we prove the classical Hölder estimates for second-order elliptic equations using a perturbation approach. For the sake of simplicity, we consider the Dirichlet boundary value problem

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3.71}$$

where

$$Lu = -\sum_{i,j=1}^{n} D_j\left(a^{ij}(x)D_i u\right) + c(x)u.$$

Recall that $u \in H_0^1(U)$ is a weak solution of (3.71) if

$$\int_U a^{ij}(x)D_i u D_j\varphi + c(x)u\varphi\,dx = \int_U f(x)\varphi\,dx \;\text{ for all }\; \varphi \in H_0^1(U).$$

As before, we assume L is uniformly elliptic, $a^{ij} \in L^\infty(U)$, the coefficient $c \in L^{\frac{n}{2}}(U)$, and $f \in L^{\frac{2n}{n+2}}(U)$. Note that the assumptions on c and f and the Sobolev embedding allows for the weak solution definition to make sense. Now, the proper space to study the Hölder regularity properties in this perturbation framework are the Morrey and Campanato spaces.

3.6.1 Morrey–Campanato Spaces

Here, we shall provide the definitions and basic properties of certain subspaces of L^p spaces—the Morrey and Campanato spaces. These function spaces allow us to generalize the Sobolev inequalities and provide the proper setting for studying the Hölder regularity of weak solutions to elliptic equations. As usual, we let $U \subset \mathbb{R}^n$ be open (not necessarily bounded) and let $U_r(x) := B_r(x) \cap U$.

Definition 3.6 (Morrey Space)**.** *Let $1 \leq p < \infty$ and $\lambda \geq 0$. The Morrey space $M^{p,\lambda}(U)$ is defined as*

$$M^{p,\lambda}(U) := \left\{ f \in L^p(U) \,\middle|\, \int_{U_r(x_0)} |f|^p \, dx \leq C^p \cdot r^\lambda \quad \text{for any} \quad x_0 \in U, \, r > 0 \right\}$$

with norm

$$\|f\|_{M^{p,\lambda}(U)} := \left(\sup_{x_0 \in U, r > 0} \frac{1}{r^\lambda} \int_{U_r(x_0)} |f|^p \, dx \right)^{1/p}.$$

Proposition 3.5. *Let $1 \leq p < \infty$ and $\lambda \geq 0$. Then*

(i) $M^{p,\lambda}(U)$ is a Banach space,

(ii) $M^{p,0}(U) = L^p(U)$,

(iii) $M^{p,n}(U) = L^\infty(U)$,

(iv) If $q > p$ then $L^q(U) \hookrightarrow M^{p,\lambda}(U)$ for $\lambda = \lambda(p,q)$.

Definition 3.7 (Type A domains)**.** *A domain U is of type A if there exists a constant $A > 0$ such that for any $x_0 \in U$ and $0 < r < diam(U)$, $|U_r(x_0)| \geq A \cdot r^n$.*

Definition 3.8 (Campanato Space)**.** *Let $1 \leq p < \infty$ and $\lambda \geq 0$. The Campanato space $L^{p,\lambda}(U)$ is defined as*

$$L^{p,\lambda}(U) := \left\{ f \in L^p(U) \,\middle|\, [f]_{L^{p,\lambda}(U)} < \infty \right\}$$

where the Campanato seminorm is given by

$$[f]_{L^{p,\lambda}(U)} := \left(\sup_{x_0 \in U, r > 0} \frac{1}{r^\lambda} \int_{U_r(x_0)} |f - (f)_{x_0,r}|^p \, dx \right)^{1/p}.$$

Remark 3.9. *Indeed, the quantity $[f]_{L^{p,\lambda}(U)}$ is a seminorm as any constant function f satisfies $[f]_{L^{p,\lambda}(U)} = 0$.*

Proposition 3.6. *Let $1 \leq p < \infty$ and $\lambda \geq 0$. Then*

(i) If U is of type A and $0 < \lambda < n$, then $M^{p,\lambda}(U) = L^{p,\lambda}(U)$,

(ii) *If* $\lambda = n$ *and* $p = 1$*, then* $L^{1,n}(U) = BMO(U)$ *for any* U*,*

(iii) *If* $\lambda > n + p$*, then for any* U *and any* p*,* $L^{p,\lambda}(U)$ *is trivial in that it only contains constant functions.*

Remark 3.10. *To summarize, the Morrey and Campanato spaces are indistinguishable in the range* $\lambda \in (0, n)$*. In the endpoint case* $p = 1$ *and* $\lambda = n$*, the Campanato space reduces to the space BMO, which is larger and properly contains the space* $L^\infty(U) = M^{p,n}(U)$*. In the interval* $\lambda \in (n, n+p]$ *we shall see that the Campanato spaces are indistinguishable from the Hölder and Lipschitz spaces, and this is precisely the setting for studying the Hölder regularity of weak solutions to elliptic equations. Of course, when* $\lambda > n+p$*, the Campanato spaces (just as with the* $C^\alpha(U)$ *spaces when* $\alpha > 1$*) are trivial consisting of only the constant functions.*

We start with the following important embedding property.

Theorem 3.19 (Sobolev–Morrey Embedding)**.** *Let* $U \subset \mathbb{R}^n$ *be of type A,* $1 \leq p < \infty$ *and* $\alpha \in (0, 1)$*. If* $u \in W^{1,p}(U)$ *such that* $Du \in L^{p,n-p+p\alpha}(U)$*, then* $u \in C^\alpha(U)$*.*

Notice that this is a generalization of Morrey's inequality and Theorem A.16; that is, we recover Theorem A.16 from this if $p > n$ and $\alpha = 1 - n/p$ (or $n - p + p\alpha = 0$). Now to prove Theorem 3.19, we will need the next result, which indicates that the Campanato space $L^{p,\lambda}(U)$ is equivalent to the Hölder space $C^\alpha(U)$ for $1 \leq p < \infty$ and $\lambda = n + p\alpha$ with $\alpha \in (0, 1)$. Indeed, this illustrates an important application of the Morrey–Campanato spaces when studying the Hölder regularity of weak solutions to elliptic equations.

Theorem 3.20. *Suppose the domain* $U \subset \mathbb{R}^n$ *is of type A and let* $\alpha \in (0, 1)$*, then*

$$L^{p,n+p\alpha}(U) = C^\alpha(U).$$

Proof. First, we prove that $C^\alpha(U) \hookrightarrow L^{p,n+p\alpha}(U)$. Observe that

$$|f(x) - (f)_{x,r}| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(y)|\, dy \leq C r^\alpha [f]_{C^\alpha(U)}.$$

Thus,

$$\begin{aligned}
\frac{1}{r^{n+p\alpha}} \int_{B_r(x_0)} |f(x) - (f)_{x_0,r}|^p\, dx &\leq \frac{1}{r^{n+p\alpha}} \int_{B_r(x_0)} |f(x) - f(x_0) + f(x_0) - (f)_{x_0,r}|^p\, dx \\
&\leq \frac{C}{r^{n+p\alpha}} \int_{B_r(x_0)} \Big(\frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha} + [f]_{C^\alpha(U)} \Big)^p |x - x_0|^{p\alpha}\, dx \\
&\leq \frac{C[f]^p_{C^\alpha(U)}}{r^{n+p\alpha}} \int_{B_r(x_0)} |x - x_0|^{p\alpha}\, dx \\
&\leq C[f]^p_{C^\alpha(U)} r^{-n-p\alpha} \int_0^r t^{n+p\alpha} \frac{dt}{t} \\
&\leq C[f]^p_{C^\alpha(U)}.
\end{aligned}$$

This implies that

$$\|f\|_{L^{p,n+p\alpha}(U)} \leq C\|f\|_{C^\alpha(U)},$$

and so $f \in L^{p,n+p\alpha}(U)$ whenever $f \in C^\alpha(U)$, i.e., $C^\alpha(U) \hookrightarrow L^{p,n+p\alpha}(U)$. Hence, it only remains to prove that $L^{p,n+p\alpha}(U) \hookrightarrow C^\alpha(U)$. For simplicity we only give the proof of this for the case $p = 2$ (see Theorem 3.21 below), since our Hölder regularity results only considers weak solutions belonging to $H^1(U) = W^{1,p=2}(U)$. □

Proof of Theorem 3.19. This clearly follows from Theorem 3.20 and Poincaré's inequality. □

3.6.2 Preliminary Estimates

The following basically states and proves special cases of Theorems 3.19 and 3.20.

Theorem 3.21. *Suppose $u \in L^2(U)$ satisfies*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \, dx \leq M^2 r^{n+2\alpha} \text{ for any } B_r(x) \subset U$$

for some $\alpha \in (0,1)$. Then $u \in C^\alpha(U)$ and for any $U' \subset\subset U$ there holds

$$\|u\|_{C^\alpha(U')} \leq C(M + \|u\|_{L^2}),$$

where $C = C(n, \alpha, U', U)$ and $\|u\|_{C^\alpha(U')} := \sup_{U'} |u| + \sup_{x,y \in U',\, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$.

Proof. **Uniform Estimate:** Denote $R_0 = dist(U', \partial U)$. For any $x_0 \in U'$ and $0 < r_1 < r_2 \leq R_0$, we have

$$\begin{aligned} |u_{x_0,r_1} - u_{x_0,r_2}|^2 &\leq (|u(x) - u_{x_0,r_1}| + |u(x) - u_{x_0,r_2}|)^2 \\ &\leq |u(x) - u_{x_0,r_1}|^2 + 2|u(x) - u_{x_0,r_1}||u(x) - u_{x_0,r_2}| + |u(x) - u_{x_0,r_2}|^2 \\ &\leq 2(|u(x) - u_{x_0,r_1}|^2 + |u(x) - u_{x_0,r_1}|^2), \end{aligned}$$

where we applied Young's inequality: $2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}$. Integrating this with respect to x in $B_{r_1}(x_0)$ yields

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \cdot \frac{\omega_n}{n} r_1^n = 2\Big\{ \int_{B_{r_1}(x_0)} |u - u_{x_0,r_1}|^2 \, dx + \int_{B_{r_2}(x_0)} |u - u_{x_0,r_2}|^2 \, dx \Big\},$$

from which the estimate

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq C(n) M^2 r_1^{-n} \left(r_1^{n+2\alpha} + r_2^{n+2\alpha} \right) \tag{3.72}$$

follows. For any $R \leq R_0$, with $r_1 = R/2^{i+1}$, $r_2 = R/2^i$, we obtain

$$|u_{x_0,2^{-(i+1)}R} - u_{x_0,2^{-i}R}| \leq C(n) 2^{-(i+1)\alpha} M R^\alpha.$$

Thus, for any $h < k$,

$$|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}| \leq \frac{C(n)}{2^{(h+1)\alpha}} M R^\alpha \sum_{i=0}^{k-h-1} 2^{-i\alpha} \leq \frac{C(n,\alpha)}{2^{h\alpha}} M R^\alpha.$$

This shows that $\{u_{x_0,2^{-i}R}\} \subset \mathbb{R}$ is a Cauchy sequence, and therefore convergent whose limit $\bar{u}(x_0)$ is independent of the choice of R, since (3.72) can be applied with $r_1 = 2^{-i}R$ and $r_2 = 2^{-i}\bar{R}$ whenever $0 < R < \bar{R} \leq R_0$. Thus, we obtain

$$\bar{u}(x_0) = \lim_{r \longrightarrow 0} u_{x_0,r} \quad \text{and} \quad |u_{x_0,r} - \bar{u}(x_0)| \leq C(n) M r^\alpha \tag{3.73}$$

for any $0 < r \leq R_0$. Recall that by Lebesgue's differentiation theorem, $\{u_{x,r}\}$ converges to u in $L^1(U)$ as $r \longrightarrow 0$, so we have $u = \bar{u}$ a.e. and the inequality in (3.73) implies $\{u_{x,r}\}$ converges uniformly to $u(x)$ in U'. Moreover, since $x \mapsto u_{x,r}$ is continuous for any $r > 0$, $u(x)$ is continuous. Again, by the estimate in (3.73), we get

$$|u(x)| \leq CMR^\alpha + |u_{x,R}|$$

for any $x \in U'$ and $R \leq R_0$. Hence, u is bounded in U' where

$$\|u\|_{L^\infty(U')} \leq C\left(MR_0^\alpha + \|u\|_{L^2(U)}\right).$$

Hölder Estimate: Let $x, y \in U'$ with $R = |x-y| < R_0/2$. Then we have

$$|u(x) - u(y)| \leq |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms are estimated by the inequality in (3.73). For the last term, we rewrite it

$$|u_{x,2R} - u_{y,2R}| \leq |u_{x,2R} - u(\zeta)| + |u_{y,2R} - u(\zeta)|,$$

and integrating with respect to ζ over $B_{2R}(x) \cap B_{2R}(y)$, which contains $B_R(x)$, yields

$$\begin{aligned} |u_{x,2R} - u_{y,2R}|^2 &\leq \frac{2}{|B_R(x)|}\left\{\int_{B_{2R}(x)} |u - u_{x,2R}|^2\,dx + \int_{B_{2R}(y)} |u - u_{y,2R}|^2\,dx\right\} \\ &\leq C(n,\alpha) M^2 R^{2\alpha}. \end{aligned}$$

Hence,

$$|u(x) - u(y)| \leq C(n,\alpha) M |x-y|^\alpha.$$

For $|x - y| > R_0/2$ we obtain

$$|u(x) - u(y)| \leq 2\|u\|_{L^\infty(U')} \leq C\left\{M + \frac{1}{R_0^\alpha}\|u\|_{L^2}\right\}|x-y|^\alpha.$$

This completes the proof. $\square$

As remarked earlier, a consequence of this result is a special case of Theorem 3.19.

Corollary 3.2. *Suppose $u \in H^1_{loc}(U)$ satisfies*

$$\int_{B_r(x)} |Du|^2\,dx \le M^2 r^{n-2+2\alpha} \ \text{ for any } \ B_r(x) \subset U$$

for some $\alpha \in (0,1)$. Then $u \in C^\alpha(U)$ and for any $U' \subset\subset U$ there holds

$$\|u\|_{C^\alpha(U')} \le C(M + \|u\|_{L^2}),$$

where $C = C(n, \alpha, U', U)$

Proof. From Poincaré's inequality, we have

$$\int_{B_r(x)} |u - u_{x,r}|^2\,dx \le C(n) r^2 \int_{B_r(x)} |Du|^2\,dx \le C_n M^2 r^{n+2\alpha},$$

and the result follows immediately from the previous theorem. □

3.6.3 Hölder Continuity of Weak Solutions

First, we state two lemmas, which are key to establishing the Hölder continuity of weak solutions. The estimates in the resulting regularity theorems in this section are sometimes called Cordes-Nirenberg type estimates.

Lemma 3.5. *Let $\varphi(t)$ be a non-negative and non-decreasing function on $[0, R]$. Suppose that*

$$\varphi(\rho) \le A\Big\{ \Big(\frac{\rho}{r}\Big)^\alpha + \epsilon \Big\}\varphi(r) + Br^\beta \ \text{ for any } \ 0 < \rho \le r \le R, \tag{3.74}$$

where A, B, α, β are non-negative constants and $\beta < \alpha$. Then, for any $\gamma \in (\beta, \alpha)$, there exists a constant $\epsilon_0 = \epsilon_0(A, \alpha, \beta, \gamma)$ such that if $\epsilon < \epsilon_0$, we have for all $0 < \rho \le r \le R$

$$\varphi(\rho) \le C\Big\{ \Big(\frac{\rho}{r}\Big)^\gamma \varphi(r) + B\rho^\beta \Big\},$$

where $C = C(A, \alpha, \beta, \gamma) > 0$. In particular, we have for any $0 < r \le R$,

$$\varphi(r) \le C\Big\{ \frac{\varphi(R)}{R^\gamma} r^\gamma + Br^\beta \Big\}.$$

Proof. For $\tau \in (0,1)$ and $r \in (0, R)$, we rewrite (3.74) as

$$\varphi(\tau r) \le \tau^\gamma (1 + \epsilon\tau^{-\alpha})\varphi(r) + Br^\beta.$$

Choosing τ so that $2A\tau^\alpha = \tau^\gamma$ and assuming $\epsilon_0 \tau^{-\alpha} < 1$, we get

$$\varphi(\tau r) \le \tau^\gamma \varphi(r) + Br^\beta \text{ for each } r < R.$$

Iterating this for all positive integers k, we obtain

$$\begin{aligned}\varphi(\tau^{k+1}r) &\le \tau^{\gamma}\varphi(\tau^k r) + B\tau^{k\beta}r^{\beta} \le \tau^{(k+1)\gamma}\varphi(r) + B\tau^{k\beta}r^{\beta}\sum_{j=0}^{k}\tau^{j(\gamma-\beta)} \\ &\le \tau^{(k+1)\gamma}\varphi(r) + \frac{B\tau^{k\beta}r^{\beta}}{1-\tau^{\gamma-\beta}}.\end{aligned}$$

From this, we choose k so that $\tau^{k+2}r < \rho \le \tau^{k+1}r$ and we arrive at

$$\varphi(\rho) \le \frac{1}{\tau^{\gamma}}\Big(\frac{\rho}{r}\Big)^{\gamma}\varphi(r) + \frac{B\rho^{\beta}}{\tau^{2\beta}(1-\tau^{\gamma-\beta})}.$$

□

Lemma 3.6. *Suppose $u \in H^1(U)$ satisfies*

$$\int_{B_r(x_0)} |Du|^2\,dx \le Mr^{\mu} \ \textit{ for any } \ B_r(x_0) \subset U,$$

for some $\mu \in [0, n)$. Then for any $U' \subset\subset U$ there holds for any $B_r(x_0) \subset U$ with $x_0 \in U'$

$$\int_{B_r(x_0)} |u|^2\,dx \le C(n, \lambda, \mu, U, U')\left(M + \int_U |u|^2\,dx\right) r^{\lambda},$$

where $\lambda = \mu + 2$ if $\mu < n - 2$ and $\lambda \in [0, n)$ if $n - 2 \le \mu < n$.

Proof. From Poincaré's inequality,

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2\,dx \le Cr^2\int_{B_r(x_0)} |Du|^2\,dx \le c(n)Mr^{\mu+2}$$

for any $x_0 \in U'$ and $0 < r \le R_0 := dist(U', \partial U)$. Hence,

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2\,dx \le c(n)Mr^{\lambda}$$

where λ is as stated in the lemma. Then for any $0\rho < r \le R_0$, we have

$$\begin{aligned}\int_{B_\rho(x_0)} u^2\,dx &\le 2\int_{B_\rho(x_0)} |u_{x_0,r}|^2\,dx + 2\int_{B_\rho(x_0)} |u - u_{x_0,r}|^2\,dx \\ &\le c(n)\rho^n|u_{x_0,r}|^2 + 2\int_{B_r(x_0)} |u - u_{x_0,r}|^2\,dx \\ &\le c(n)\Big(\frac{\rho}{r}\Big)^n\int_{B_r(x_0)} u^2\,dx + Mr^{\lambda},\end{aligned}$$

where we used $|u_{x_0,r}|^2 \le \frac{c(n)}{r^n}\int_{B_r(x_0)} u^2\,dx$. Indeed, it follows that $\varphi(r) = \int_{B_r(x_0)} u^2\,dx$ satisfies

$$\varphi(\rho) \le c(n)\Big[\Big(\frac{\rho}{r}\Big)\varphi(r) + Mr^\lambda\Big] \text{ for any } 0 < \rho < r \le R_0.$$

Therefore, Lemma 3.5 implies that for any $0 < \rho < r \le R_0$,

$$\int_{B_\rho(x_0)} u^2\,dx \le c\Big[\Big(\frac{\rho}{r}\Big)^\lambda \int_{B_r(x_0)} u^2\,dx + M\rho^\lambda\Big].$$

In particular, if $r = R_0$,

$$\int_{B_\rho(x_0)} u^2\,dx \le c\rho^\lambda\Big(M + \int_U u^2\,dx\Big) \text{ for } 0 < \rho \le R_0.$$

□

To best illustrate the main ideas in the Hölder continuity of weak solutions, we assume that $U = B_1 = B_1(0)$.

Theorem 3.22. *Let $u \in H^1(B_1)$ be a weak solution of (3.71). Assume $a^{ij} \in C(\bar{B}_1)$, $c \in L^n(B_1)$, and $f \in L^q(B_1)$ for some $q \in (n/2, n)$. Then $u \in C^\alpha(B_1)$ with $\alpha = 2 - n/q \in (0,1)$. Moreover, there exists an $R_0 = R_0(\lambda, \Lambda, \tau, \|c\|_{L^n})$ such that for any $x \in B_{1/2}$ and $r \le R_0$, there holds*

$$\int_{B_r(x)} |Du|^2\,dx \le Cr^{n-2+2\alpha}\Big\{\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\Big\},$$

where $C = C(\lambda, \Lambda, \tau, \|c\|_{L^n})$ is a positive constant with

$$|a^{ij}(x) - a^{ij}(y)| \le \tau|x-y| \text{ for any } x, y \in B_1.$$

Remark 3.11. *In the case where $c \equiv 0$, we may replace $\|u\|_{H^1(B_1)}$ with $\|Du\|_{L^2(B_1)}$.*

The main idea in the proof is to compare the solution with harmonic functions and use a perturbation argument. So we rely on the previous estimates and comparison results on harmonic functions.

Lemma 3.7 (Basic Estimates for Harmonic Functions)**.** *Suppose $\{a^{ij}\}$ is a constant positive definite matrix satisfying the uniformly elliptic condition,*

$$\lambda|\xi|^2 \le a^{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for any } \xi \in \mathbb{R}^n$$

for some $0 < \lambda \le \Lambda$. Suppose $w \in H^1(B_r(x_0))$ is a weak solution of $D_i(a^{ij}(x)D_jw) = 0$ in $B_r(x_0)$. Then for any $0 < \rho \le r$, there hold

$$\int_{B_\rho(x_0)} |Dw|^2\,dx \le C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2\,dx,$$

$$\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2\,dx \le C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,r}|^2\,dx,$$

where $C = C(\lambda, \Lambda)$.

Proof. This follows from Lemma 1.3 with u replaced by Dw instead. □

Lemma 3.8 (Comparison with Harmonic Functions)**.** *Suppose w is as in the previous lemma. Then for any $u \in H_0^1(B_r(x_0))$ there hold for any $0 < \rho \le r$,*

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Big\{ \Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |Du|^2\,dx + \int_{B_r(x_0)} |D(u-w)|^2\,dx \Big\},$$

$$\int_{B_\rho(x_0)} |Du-(Du)_{x_0,r}|^2\,dx \le C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du-(Du)_{x_0,r}|^2\,dx + \int_{B_r(x_0)} |D(u-w)|^2\,dx \Big\},$$

where $C = C(\lambda, \Lambda)$.

Proof. We prove this by directly by simple computations. With $v = u - w$ we have that for any $0 < \rho \le r$,

$$\begin{aligned}
\int_{B_\rho(x_0)} |Du|^2\,dx &\le 2\int_{B_\rho(x_0)} |Dw|^2\,dx + 2\int_{B_\rho(x_0)} |Dv|^2\,dx \\
&\le C\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |Dw|^2\,dx + 2\int_{B_r(x_0)} |Dv|^2\,dx \\
&\le C\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |Du|^2\,dx + C\Big\{1 + \Big(\frac{\rho}{r}\Big)^n\Big\} \int_{B_r(x_0)} |Dv|^2\,dx,
\end{aligned}$$

and

$$\begin{aligned}
\int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2\,dx &\le 2\int_{B_\rho(x_0)} |Du-(Dw)_{x_0,\rho}|^2\,dx + 2\int_{B_\rho(x_0)} |Dv|^2\,dx \\
&\le 4\int_{B_\rho(x_0)} |Dw-(Dw)_{x_0,\rho}|^2\,dx + 6\int_{B_\rho(x_0)} |Dv|^2\,dx \\
&\le C\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Dw-(Dw)_{x_0,\rho}|^2\,dx + 6\int_{B_r(x_0)} |Dv|^2\,dx \\
&\le C\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du-(Du)_{x_0,\rho}|^2\,dx \\
&\qquad + C\Big\{1 + \Big(\frac{\rho}{r}\Big)^{n+2}\Big\} \int_{B_r(x_0)} |Dv|^2\,dx.
\end{aligned}$$

□

Proof of Theorem 3.22. We decompose u into a sum $v + w$ where w satisfies a homogeneous equation and v has estimates in terms of non-homogeneous terms.

For any $B_r(x_0) \subset B_1$, write the equation as

$$\int_{B_1} a^{ij}(x_0) D_i u D_j\varphi\,dx = \int_{B_1} f\varphi - cu\varphi + (a^{ij}(x_0) - a^{ij}(x)) D_i u D_j\varphi\,dx.$$

In $B_r(x_0)$, the Dirichlet problem,

$$\int_{B_r(x_0)} a^{ij}(x_0) D_i w D_j \varphi \, dx = 0 \quad \text{for any} \quad \varphi \in H_0^1(B_r(x_0))$$

has a unique weak solution in $H_0^1(B_r(x_0))$ and $u - w \in H_0^1(B_r(x_0))$. Clearly, $v = u - w$ belongs in $H_0^1(B_r(x_0))$ and satisfies

$$\int_{B_1} a^{ij}(x_0) D_i v D_j \varphi \, dx = \int_{B_1} f\varphi - cu\varphi + (a^{ij}(x_0) - a^{ij}(x)) D_i u D_j \varphi \, dx \tag{3.75}$$

for any $\varphi \in H_0^1(B_r(x_0))$. By taking the test function $\varphi = v$, we have the following estimates on each term in the right-hand side of (3.75):

$$\begin{aligned}
\int_{B_r(x_0)} fv \, dx &\leq \left(\int_{B_r(x_0)} f^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{2n}} \left(\int_{B_r(x_0)} |v|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{2n}} \\
&\leq \left(\int_{B_r(x_0)} f^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{2n}} \left(\int_{B_r(x_0)} |Dv|^2 \, dx \right)^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\int_{B_r(x_0)} cuv \, dx &\leq \left(\int_{B_r(x_0)} |c|^n \, dx \right)^{\frac{1}{n}} \left(\int_{B_r(x_0)} |uv|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \\
&\leq \left(\int_{B_r(x_0)} |c|^n \, dx \right)^{\frac{1}{n}} \left(\int_{B_r(x_0)} |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |v|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{2n}} \\
&\leq \left(\int_{B_r(x_0)} |c|^n \, dx \right)^{\frac{1}{n}} \left(\int_{B_r(x_0)} |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |Dv|^2 \, dx \right)^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\int_{B_r(x_0)} (a^{ij}(x_0) - a^{ij}(x)) D_i u D_j v \, dx &\leq \tau(r)^2 \left(\int_{B_r(x_0)} |Du|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |v|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{2n}} \\
&\leq \tau(r)^2 \left(\int_{B_r(x_0)} |Du|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |Dv|^2 \, dx \right)^{\frac{1}{2}},
\end{aligned}$$

where we used Hölder's inequality and the Sobolev embedding theorem. From the uniform ellipticity condition, we estimate the terms in (3.75) by using the previous three estimates then divide both sides of the inequality by $\|Dv\|_{L^2(B_r(x_0))}$ to get

$$\begin{aligned}
&\int_{B_r(x_0)} |Dv|^2 \, dx \\
&\quad \leq C \left\{ \tau(r)^2 \int_{B_r(x_0)} |Du|^2 \, dx + \left(\int_{B_r(x_0)} |c|^n \, dx \right)^{2/n} \int_{B_r(x_0)} |u|^2 \, dx + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \right\}.
\end{aligned}$$

Therefore, Corollary 3.8 implies that for any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right)\int_{B_r(x_0)} |Du|^2\,dx \\ + \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{2/n}\int_{B_r(x_0)} |u|^2\,dx + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}\Bigg\}, \quad (3.76)$$

where $C = (n, \lambda, \Lambda)$ is a positive constant. By Hölder's inequality,

$$\left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}} \le \left(\int_{B_r(x_0)} |f|^q\,dx\right)^{\frac{2}{q}} r^{n-2+2\alpha},$$

where $\alpha = 2 - \frac{n}{q} \in (0,1)$ whenever $q \in (\frac{n}{2}, n)$. Thus, (3.76) implies for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right)\int_{B_r(x_0)} |Du|^2\,dx \\ + \left(\int_{B_r(x_0)} |c|^n\,dx\right)^{2/n}\int_{B_r(x_0)} |u|^2\,dx + r^{n-2+2\alpha}\|f\|^2_{L^q(B_1)}\Bigg\}.$$

Case 1: $c \equiv 0$.

We have for any $B_r(x_0) \subset B_1$ and for any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right)\int_{B_r(x_0)} |Du|^2\,dx + r^{n-2+2\alpha}\|f\|^2_{L^q(B_1)}\Bigg\}.$$

By Lemma 3.5, we may replace $r^{n-2+2\alpha}$ in the last estimate by $\rho^{n-2+2\alpha}$, in which case the proof is complete. More precisely, there exists an $R_0 > 0$ such that for any $x_0 \in B_{1/2}(0)$ and any $0 < \rho < r \le R_0$, we have

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right)\int_{B_r(x_0)} |Du|^2\,dx + \rho^{n-2+2\alpha}\|f\|^2_{L^q(B_1)}\Bigg\}.$$

In particular, taking $r = R_0$ yields for any $\rho < R_0$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\rho^{n-2+2\alpha}\Bigg\{ \int_{B_1} |Du|^2\,dx + \|f\|^2_{L^q(B_1)}\Bigg\}.$$

Case 2: General coefficient $c \in L^n(B_1)$. We have for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\Bigg\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right)\int_{B_r(x_0)} |Du|^2\,dx + r^{n-2+2\alpha}\chi(F) + \int_{B_r(x_0)} u^2\,dx\Bigg\} \quad (3.77)$$

where $\chi(F) = \|f\|^2_{L^q(B_1)}$. We will prove, via a bootstrap argument, that for any $x_0 \in B_{1/2}$ and any $0 < \rho < r < 1/2$,

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\left\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx \right.$$
$$\left. + r^{n-2+2\alpha}\left(\chi(F) + \int_{B_1} u^2\,dx + \int_{B_1} |Du|^2\,dx\right)\right\}. \tag{3.78}$$

First by Lemma 3.6, there exists an $R_1 \in (1/2, 1)$ such that there holds for any $x_0 \in B_{R_1}$ and any $0 < r \le 1 - R_1$

$$\int_{B_r(x_0)} u^2\,dx \le Cr^{\delta_1}\left\{\int_{B_1} |Du|^2\,dx + \int_{B_1} u^2\,dx\right\} \tag{3.79}$$

where $\delta_1 = 2$ if $n > 2$ and δ_1 is arbitrary in $(0, 2)$ if $n = 2$. This, combined with (3.77), implies

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\left\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx + r^{n-2+2\alpha}\chi(F) + r^{\delta_1}\|u\|^2_{H^1(B_1)}\right\}.$$

Then (3.78) holds in the following cases:

(i) $n = 2$, by choosing $\delta_1 = 2\alpha$,

(ii) $n > 2$ while $n - 2 + 2\alpha \le 2$, by choosing $\delta_1 = 2$.

However, for $n > 2$ and $n - 2 + 2\alpha > 2$, we have

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le C\left\{ \left(\left(\frac{\rho}{r}\right)^n + \tau(r)^2\right) \int_{B_r(x_0)} |Du|^2\,dx + r^2\left(\chi(F) + r^{\delta_1}\|u\|^2_{H^1(B_1)}\right)\right\}.$$

□

Lemma 3.5 again implies that for any $x_0 \in B_{R_1}$ and any $0 < r \le 1 - R_1$

$$\int_{B_r(x_0)} |Du|^2\,dx \le Cr^2\left\{\chi(F) + \|u\|^2_{H^1(B_1)}\right\}.$$

Then by Lemma 3.6, there exists an $R_2 \in (1/2, R_1)$ such that there holds for any $x_0 \in B_{R_2}$ and any $0 < r \le R_1 - R_2$

$$\int_{B_r(x_0)} u^2\,dx \le Cr^{\delta_2}\left\{\chi(F) + \|u\|^2_{H^1(B_1)}\right\} \tag{3.80}$$

where $\delta_2 = 4$ if $n > 4$ and δ_2 is arbitrary in $(2, n)$ if $n = 3$ or 4. Notice that this last estimate (3.80) is an improvement compared with (3.79). Substitute (3.80) in (3.77) and continue the process. After finite steps we arrive at (3.78).

3.6.4 Hölder Continuity of the Gradient

As before, we take $U = B_1$. We have the following estimate for the gradient of weak solutions of equation (3.71). The proof is similar as before, so we omit the details.

Theorem 3.23. *Let $u \in H^1(B_1)$ be a weak solution of (3.71). Assume $a^{ij} \in C^\alpha(\bar{B}_1)$, $c \in L^q(B_1)$ and $f \in L^q(B_1)$ for some $q > n$ and $\alpha = 1 - n/q \in (0,1)$. Then $Du \in C^\alpha(B_1)$. Moreover, there exists an $R_0 = R_0(\lambda, |a^{ij}|_{C^\alpha}, \|c\|_{L^q})$ such that for any $x \in B_{1/2}$ and $r \leq R_0$, there holds*

$$\int_{B_r(x)} |Du - (Du)_{x,r}|^2 \, dx \leq C r^{n+2\alpha} \Big\{ \|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)} \Big\},$$

where $C = C(\lambda, |a^{ij}|_{C^\alpha}, \|c\|_{L^q})$ is a positive constant.

3.7 De Giorgi–Nash–Moser Regularity Theory

This section introduces the celebrated De Giorgi–Nash–Moser regularity theory for the Hölder continuity of solutions, and we introduce two ideas for completeness. That is, we first introduce De Giorgi's approach which develops the local boundedness of solutions followed by the estimate on its oscillation. These two ingredients will imply the Hölder continuity of solutions. Then, we study Moser's approach, which also uses the same local boundedness result combined with Moser's version of the Harnack inequality to conclude the same result on the Hölder continuity of solutions. Note carefully that, unlike in the previous section, we will not make any regularity assumptions on the coefficients of the elliptic operators. Furthermore, the overall idea we use here relies on a delicate iteration technique rather than perturbation methods.

3.7.1 Motivation

Before we proceed with the technical aspects of this theory, let us motivate its historical relevance. Recall that the nineteenth problem in Hilbert's famous program asked whether or not minimizers $w \in H^1_0(U) \cap H^2(U)$ of the energy functional

$$J(w) = \int_U L(Dw) \, dx$$

are smooth. The Lagrangian L is assumed to be smooth and satisfies some additional conditions (such as those described in Chapter 2, specifically the first section on the calculus of variations). The Euler-Lagrange equation for this variational problem is the nonlinear elliptic equation

$$\sum_{i=1}^n (L_{p_i}(Dw))_{x_i} = 0 \;\text{ in } U. \tag{3.81}$$

Indeed, the minimizers are smooth and this can be proved using the Schauder estimates and a standard bootstrap argument. To carry out the procedure, however, we initially require the minimizer to be of class $C^{1,\alpha}$. The main result of the De Giorgi–Nash–Moser theory precisely ensures minimizers are of class $C^{1,\alpha}$ and thus it provided the crucial ingredient in resolving Hilbert's nineteenth problem.

What follows is only a rough explanation of the procedure but the arguments can certainly be made rigorous. If we formally differentiate equation (3.81) with respect to x_k then insert (3.81) into the resulting calculation, we would obtain

$$\sum_{i,j=1}^{n}(L_{p_ip_j}(Dw)w_{x_jx_k})_{x_i} = 0.$$

Thus, if we set $u = w_{x_k}$, this implies that u satisfies the linear elliptic equation

$$\sum_{i,j=1}^{n}(a^{ij}(x)u_{x_j})_{x_i} = 0, \tag{3.82}$$

where $a^{ij}(x) = L_{p_ip_j}(Dw(x))$ satisfies some type of uniform ellipticity condition. De Giorgi–Nash–Moser theory ensures that if u is a weak solution of equation (3.82), then u is Hölder continuous and so w is a $C^{1,\alpha}$ solution of (3.81). Hence, the coefficients $a^{ij}(x)$ are Hölder continuous and the Schauder estimates imply that $u \in C^{2,\alpha}$. By bootstrap, u is of class $C^{k,\alpha}$ for $k = 2, 3, 4, \ldots$ and is therefore, along with w, smooth.

3.7.2 Local Boundedness and Preliminary Lemmas

Both De Giorgi and Moser's approach rely initially on the local boundedness of solutions before arriving at the Hölder regularity result. We now state this result but defer its proof to the next section.

Theorem 3.24 (local boundedness). *Suppose $a^{ij} \in L^\infty(B_1)$ and $c \in L^q(B_1)$ for some $q > n/2$ satisfy the following assumptions:*

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \ \text{ for any } \ x \in B_1,\ \xi \in \mathbb{R}^n,$$

and

$$\|a^{ij}\|_{L^\infty(B_1)} + \|c\|_{L^q(B_1)} \leq \Lambda$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a sub-solution in the following sense:

$$\int_{B_1} a^{ij}D_iuD_j\varphi + cu\varphi\,dx \leq \int_{B_1} f\varphi\,dx \ \text{ for any non-negative } \varphi \in H_0^1(B_1). \tag{3.83}$$

If $f \in L^q(B_1)$, then $u^+ \in L^\infty_{loc}(B_1)$. Moreover, there holds for any $\theta \in (0,1)$ and $p > 0$

$$\sup_{B_\theta} u^+ \leq C\Big\{ \frac{1}{(1-\theta)^{n/p}} \|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)} \Big\},$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

One strategy to prove this is to use a clever iteration procedure of Moser, which will also appear in our proof of the weak Harnack inequality below. In either case, Moser's iteration procedure will also make use of the following elementary result.

Lemma 3.9. *Let U be a bounded subset, $u : U \mapsto \mathbb{R}$ is measurable, $|u|^p \in L^1(U)$ for $p \geq 1$ and assume*

$$\Phi(p) := \Big(\frac{1}{|U|} \int_U |u|^p \, dx \Big)^{1/p}$$

is well-defined. Then

$$\lim_{p \to \infty} \Phi(p) = \sup_U u.$$

Proof. Assume $p' > p$ is arbitrary. If $u \in L^{p'}(U)$, then Hölder's inequality yields

$$\begin{aligned} \Big(\frac{1}{|U|} \int_U u^p \, dx \Big)^{1/p} &\leq \frac{1}{|U|^{1/p}} \Big(\int_U 1 \, dx \Big)^{\frac{p'-p}{pp'}} \Big(\int_U (u^p)^{p'/p} \, dx \Big)^{1/p'} \\ &= \Big(\frac{1}{|U|} \int_U u^{p'} \, dx \Big)^{1/p'}. \end{aligned}$$

Hence, $\Phi(p)$ is monotone increasing with respect to $p > 1$. Moreover, $\Phi(p)$ is bounded above by $\sup_U u$ since

$$\Phi(p) \leq \Big(\frac{1}{|U|} \int_U (\sup_U u)^p \, dx \Big)^{1/p} \leq \sup_U u.$$

Thus, $\lim_{p \to \infty} \Phi(p) \leq \sup_U u$.

On the other hand, by definition of the essential supremum, for each $\epsilon > 0$ there exists $\delta > 0$ such that $|A| > \delta$, where

$$A = \{ x \in U \,|\, u(x) \geq \sup_U u - \epsilon \}.$$

Therefore,

$$\Phi(p) \geq \Big(\frac{1}{|A|} \int_A u^p \, dx \Big)^{1/p} \geq \sup_U u - \epsilon.$$

Hence, for every $\epsilon > 0$,

$$\lim_{p \to \infty} \Phi(p) \geq \sup u - \epsilon,$$

which immediately implies that $\lim_{p \to \infty} \Phi(p) \geq \sup u$. This completes the proof of the lemma. $\square$

After establishing local boundedness, the Hölder continuity of weak solutions will be a consequence of the following important lemma and a Harnack or oscillation inequality.

Lemma 3.10. *Let ω and σ be non-decreasing functions in an interval $(0, R]$. Suppose there holds for all $r \leq R$,*

$$\omega(\tau r) \leq \gamma \omega(r) + \sigma(r)$$

for some $0 < \gamma, \tau < 1$. Then for any $\mu \in (0,1)$ and $r \leq R$ we have

$$\omega(r) \leq C\Big\{\Big(\frac{r}{R}\Big)^{\alpha} \omega(R) + \sigma(r^{\mu} R^{1-\mu})\Big\}$$

where $C = C(\gamma, \tau)$ is a positive constant and $\alpha = (1-\mu)\log\gamma/\log\tau$.

Proof. Fix some $r_1 \leq R$. Then for any $r \leq r_1$ we have

$$\omega(\tau r) \leq \gamma \omega(r) + \sigma(r_1)$$

since σ is non-decreasing. We now iterate this inequality to get for any positive integer k

$$\omega(\tau^k r_1) \leq \gamma^k \omega(r_1) + \sigma(r_1) \sum_{i=0}^{k-1} \gamma^i \leq \gamma^k \omega(R) + \frac{\sigma(r_1)}{1-\gamma}.$$

For any $r \leq r_1$, choose k so that

$$\tau^k r_1 < r \leq \tau^{k-1} r_1.$$

This ensures that $(\log \gamma^k)(\log \tau) \leq (\log \gamma)(\log(r/r_1))$ and so

$$\gamma^k \leq (r/r_1)^{\log\gamma/\log\tau}.$$

Hence, the monotonicity of ω then implies that

$$\omega(r) \leq \omega(\tau^{k-1} r_1) \leq \gamma^{k-1}\omega(R) + \frac{\sigma(r_1)}{1-\gamma} \leq \frac{1}{\gamma}\Big(\frac{r}{r_1}\Big)^{\frac{\log\gamma}{\log\tau}} \omega(R) + \frac{\sigma(r_1)}{1-\gamma}.$$

If we take $r_1 = r^{\mu} R^{1-\mu}$, we obtain

$$\omega(r) \leq \frac{1}{\gamma}\Big(\frac{r}{R}\Big)^{(1-\mu)\frac{\log\gamma}{\log\tau}} \omega(R) + \frac{\sigma(r^{\mu}R^{1-\mu})}{1-\gamma}.$$

$\square$

3.7.3 Proof of Local Boundedness: Moser Iteration

To illustrate the main idea in our proof of Theorem 3.24, let us describe our strategy for the case when $f \equiv 0$, $\theta = 1/2$ and $p = 2$. By choosing an appropriate test function, we will estimate the L^{p_1} norm of u in a smaller ball by the L^{p_2} norm of u in a larger ball for $p_1 > p_2$; that is, we establish a reverse type Hölder inequality

$$\|u\|_{L^{p_1}(B_{r_1})} \leq C\|u\|_{L^{p_2}(B_{r_2})}, \tag{3.84}$$

for $p_1 > p_2$ and $r_1 < r_2$. The issue is our choice of test function forces the constant C to behave like $(r_2 - r_1)^{-1}$. Moser's approach, however, is to carefully iterate the estimate and choose sequences $\{r_i\}$ and $\{p_i\}$ which avoids this constant from blowing up. Thus, this iteration technique and Lemma 3.9 allows us to send $p_1 \to \infty$, $p_2 \to 2$, $r_1 \to 1/2$ and $r_2 \to 1$ in (3.84) to get the desired estimate.

Proof of Theorem 3.24. The proof is long, so we divide it into several steps.

Step 1: We prove the theorem for $\theta = 1/2$ and $p = 2$. We follow Moser's proof, but an alternative proof by De Giorgi can also be found in [13]. For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m, \\ m + k & \text{if } u \geq m. \end{cases}$$

Then we have $D\bar{u}_m = 0$ in $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \leq \bar{u}$. Set the test function

$$\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1)$$

for some $\beta \geq 0$ and some non-negative function $\eta \in C_0^1(B_1)$. Direct calculation yields

$$\begin{aligned} D\varphi &= \beta\eta^2\bar{u}_m^{\beta-1}D\bar{u}_m\bar{u} + \eta^2\bar{u}_m^\beta D\bar{u} + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}) \\ &\geq \eta^2\bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}). \end{aligned} \tag{3.85}$$

Note that $\varphi = 0$ and $D\varphi = 0$ in $\{u \leq 0\}$. Hence, if we substitute such φ in the equation, we integrate in the set $\{u > 0\}$ then send m to infinity. Note also that $u^+ \leq \bar{u}$ and $\bar{u}_m^\beta\bar{u} - k^{\beta+1} \leq \bar{u}_m^\beta\bar{u}$ for $k > 0$. From the elementary inequality $ab \leq 2ab \leq a^2 + b^2$ for $a, b \geq 0$, we have

$$\begin{aligned} \Lambda|D\bar{u}||D\eta|\bar{u}_m^\beta\bar{u}\eta &= a \times b \\ &:= \Lambda(2/\lambda)^{1/2}|D\eta|\bar{u}_m^{\beta/2}\bar{u} \times (\lambda/2)^{1/2}\eta\bar{u}_m^{\beta/2}|D\bar{u}| \\ &\leq \frac{2\Lambda^2}{\lambda}|D\eta|^2\bar{u}_m^\beta\bar{u}^2 + \frac{\lambda}{2}\eta^2\bar{u}_m^\beta|D\bar{u}|^2. \end{aligned} \tag{3.86}$$

Hence,

$$\begin{aligned}\int a^{ij}(x)D_iuD_j\varphi\,dx &= \int a^{ij}(x)D_i\bar{u}(\beta D_j\bar{u}_m + D_j\bar{u})\eta^2\bar{u}_m^\beta + 2\int a^{ij}(x)D_i\bar{u}D_j\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1})\eta\,dx\\
&\geq \lambda\beta\int \eta^2\bar{u}_m^\beta|D\bar{u}_m|^2\,dx + \lambda\int\eta^2\bar{u}_m^\beta|D\bar{u}|^2\,dx - \Lambda\int|D\bar{u}||D\eta|\bar{u}_m^\beta\bar{u}\eta\,dx\\
&\geq \lambda\beta\int \eta^2\bar{u}_m^\beta|D\bar{u}_m|^2\,dx + \frac{\lambda}{2}\int\eta^2\bar{u}_m^\beta|D\bar{u}|^2\,dx - \frac{2\Lambda^2}{\lambda}\int|D\eta|^2\bar{u}_m^\beta\bar{u}^2\,dx,\end{aligned}$$

where we used (3.85) in the first line and we used (3.86) to estimate the last line. Therefore, noting that $\bar{u}\geq k$, we obtain

$$\begin{aligned}\beta\int\eta^2\bar{u}_m^\beta|D\bar{u}_m|^2\,dx + \int\eta^2\bar{u}_m^\beta|D\bar{u}|^2\,dx &\leq C\left\{\int|D\eta|^2\bar{u}_m^\beta\bar{u}^2\,dx + \int|c|\eta^2\bar{u}_m^\beta\bar{u}^2 + |f|\eta^2\bar{u}_m^\beta\bar{u}\,dx\right\}\\
&\leq C\left\{\int|D\eta|^2\bar{u}_m^\beta\bar{u}^2\,dx + \int c_0\eta^2\bar{u}_m^\beta\bar{u}^2\,dx\right\}, \quad (3.87)\end{aligned}$$

where c_0 is defined as

$$c_0 = |c| + \frac{|f|}{k}.$$

Choose $k = \|f\|_{L^q(B_1)}$ if f is not identically 0. Otherwise, choose arbitrary $k > 0$ and eventually let $k \longrightarrow 0^+$. By assumption, we have

$$\|c_0\|_{L^q} \leq \Lambda + 1.$$

Set $w = \bar{u}_m^{\beta/2}\bar{u}$ and so

$$|Dw|^2 \leq (1+\beta)\left\{\beta\bar{u}_m^\beta|D\bar{u}_m|^2 + \bar{u}_m^\beta|D\bar{u}|^2\right\}.$$

Thus, from (3.87) we have

$$\int|Dw|^2\eta^2\,dx \leq C\left\{(1+\beta)\int w^2|D\eta|^2\,dx + (1+\beta)\int c_0w^2\eta^2\,dx\right\}$$

or

$$\int|D(w\eta)|^2\eta^2\,dx \leq C\left\{(1+\beta)\int w^2|D\eta|^2\,dx + (1+\beta)\int c_0w^2\eta^2\,dx\right\}. \qquad (3.88)$$

Hölder's inequality implies

$$\int c_0w^2\eta^2\,dx \leq \left(\int c_0^q\,dx\right)^{\frac{1}{q}}\left(\int(\eta w)^{\frac{2q}{q-1}}\,dx\right)^{1-1/q} \leq (1+\Lambda)\left(\int(\eta w)^{\frac{2q}{q-1}}\,dx\right)^{1-1/q}.$$

By interpolation and Sobolev embedding with $2^* = \frac{2n}{n-2} > \frac{2q}{q-1} > 2$ if $q > \frac{n}{2}$, we have

$$\begin{aligned}\|\eta w\|_{L^{\frac{2q}{q-1}}} &\le \epsilon\|\eta w\|_{L^{2^*}} + C(n,q)\epsilon^{-\frac{n}{2q-n}}\|\eta w\|_{L^2} \\ &\le \epsilon\|D(\eta w)\|_{L^2} + C(n,q)\epsilon^{-\frac{n}{2q-n}}\|\eta w\|_{L^2}\end{aligned}$$

for small $\epsilon > 0$. Therefore, combining this with (3.88) yields

$$\int |D(w\eta)|^2\,dx \le C\left\{(1+\beta)\int w^2|D\eta|^2\,dx + (1+\beta)^{\frac{2q}{2q-n}}\int w^2\eta^2\,dx\right\},$$

and in particular

$$\int |D(w\eta)|^2\,dx \le C(1+\beta)^\alpha \int (|D\eta|^2+\eta^2)w^2\,dx,$$

where α is a positive number depending only on n and q. Sobolev embedding then implies

$$\left(\int |\eta w|^{2\chi}\,dx\right)^{1/\chi} \le C(1+\beta)^\alpha \int (|D\eta|^2+\eta^2)w^2\,dx,$$

where $\chi = \frac{n}{n-2} > 1$ for $n > 2$ and $\chi > 2$ for $n = 2$.

Choose the cutoff function as follows. For any $0 < r < R \le 1$, set $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_r \text{ and } |D\eta| \le \frac{2}{R-r}.$$

Then we obtain

$$\left(\int_{B_r} w^{2\chi}\,dx\right)^{1/\chi} \le C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} w^2\,dx.$$

If we recall the definition of w, we have

$$\left(\int_{B_r} \bar{u}^{2\chi}\bar{u}_m^{\beta\chi}\,dx\right)^{1/\chi} \le C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^2\bar{u}_m^\beta\,dx.$$

Set $\gamma = \beta + 2 \ge 2$, then we get

$$\left(\int_{B_r} \bar{u}_m^{\gamma\chi}\,dx\right)^{1/\chi} \le C\frac{(\gamma-1)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^\gamma\,dx$$

provided that the integral on the right-hand side is finite. By sending $m \longrightarrow \infty$, we conclude that

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \le \left(C\frac{(\gamma-1)^\alpha}{(R-r)^2}\right)^{1/\gamma}\|\bar{u}\|_{L^\gamma(B_R)}$$

provided that $\|\bar{u}\|_{L^\gamma(B_R)} < \infty$, where $C = C(n, q, \lambda, \Lambda)$ is a positive constant independent of γ. We shall iterate the previous estimated beginning with $\gamma = 2$ and proceed via $2, 2\chi, 2\chi^2, \dots$. Now set for $i = 0, 1, 2, \dots,$

$$\gamma_i = 2\chi^i \text{ and } r_i = \frac{1}{2} + \frac{1}{2^{i+1}}.$$

Since $\gamma_i = \chi\gamma_{i-1}$ and $r_{i-1} - r_i = 2^{-(i+1)}$, we have for $i = 1, 2, \dots,$

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C(n, q, \lambda, \Lambda)^{\frac{i}{\chi^i}} \|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})}$$

provided that $\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_i-1})} < \infty$. Hence, by iteration, we obtain

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C^{\sum \frac{i}{\chi^i}} \|\bar{u}\|_{L^2(B_1)}$$

or in particular,

$$\left(\int_{B_{1/2}} \bar{u}^{2\chi^i} \, dx \right)^{\frac{1}{2\chi^i}} \le C \left(\int_{B_1} \bar{u}^2 \, dx \right)^{\frac{1}{2}}.$$

Sending $i \longrightarrow \infty$ in the previous esitmate yields

$$\sup_{B_{1/2}} \bar{u} \le C\|\bar{u}\|_{L^2(B_1)} \text{ or } \sup_{B_{1/2}} u^+ \le C\|u^+\|_{L^2(B_1)} + k = C\Big\{\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\}.$$

This completes the proof of the theorem for the case $p = 2$.

Remark 3.12. *If the subsolution u is bounded, we may simply take the test function*

$$\varphi = \eta^2(\bar{u}^{\beta+1} - k^{\beta+1}) \in H_0^1(B_1).$$

for some $\beta \ge 0$ and some non-negative function $\eta \in C_0^1(B_1)$.

Step 2: We now prove the theorem for $p \ge 2$.

Based on a dilation argument, we take any $R \le 1$ and define

$$\tilde{u}(y) = u(Ry) \text{ for } y \in B_1.$$

It is easy to see that $\tilde{u}$ satisfies

$$\int_{B_1} \tilde{a}^{ij}(x) D_i\tilde{u}\, D_j\varphi + \tilde{c}\tilde{u}\varphi \, dx \le \int_{B_1} \tilde{f}\varphi \, dx$$

for any non-negative $\varphi \in H_0^1(B_1)$ where

$$\tilde{a}(y) = a(Ry),\ \tilde{c}(y) = R^2 c(Ry), \text{ and } \tilde{f}(y) = R^2 f(Ry),$$

for any $y \in B_1$. Direct calculation shows

$$\|\tilde{a}^{ij}\|_{L^\infty(B_1)} + \|\tilde{c}\|_{L^q(B_1)} = \|\tilde{a}^{ij}\|_{L^\infty(B_1)} + R^{2-n/q}\|c\|_{L^q(B_R)} \leq \Lambda.$$

We may apply what we proved above to $\tilde{u}$ in B_1 (iterating with $\gamma = p$ instead of $\gamma = 2$) and rewrite the result in terms of u. Hence, we obtain for $p \geq 2$

$$\sup_{B_{R/2}} u^+ \leq C\left\{R^{-n/p}\|u^+\|_{L^p(B_R)} + R^{2-n/q}\|f\|_{L^q(B_R)}\right\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant. The estimate in $B_{\theta R}$ can be obtained by applying the above result to $B_{(1-\theta)R}(y)$ for any $y \in B_{\theta R}$. Take $R = 1$. This is Theorem 3.24 for any $\theta \in (0,1)$ and $p \geq 2$.

Step 3: We now prove the theorem for $p \in (0,2)$. We show that for any $\theta \in (0,1)$ and $0 < R \leq 1$ there holds

$$\begin{aligned}\|u^+\|_{L^\infty(B_{\theta R})} &\leq C\left\{\frac{1}{[(1-\theta)R]^{n/2}}\|u^+\|_{L^2(B_R)} + R^{2-n/q}\|f\|_{L^q(B_R)}\right\}\\ &\leq C\left\{\frac{1}{[(1-\theta)R]^{n/2}}\|u^+\|_{L^2(B_R)} + \|f\|_{L^q(B_R)}\right\}.\end{aligned}$$

For $p \in (0,2)$ we have

$$\int_{B_R} (u^+)^2\, dx \leq \|u^+\|^{2-p}_{L^\infty(B_R)} \int_{B_R} (u^+)^p\, dx.$$

Thus, by Hölder's inequality,

$$\begin{aligned}\|u^+\|_{L^\infty(B_{\theta R})} &\leq C\left\{\frac{1}{[(1-\theta)R]^{n/2}}\|u^+\|^{1-p/2}_{L^\infty(B_R)}\left(\int_{B_R}(u^+)^p\, dx\right)^{\frac{1}{2}} + \|f\|_{L^q(B_R)}\right\}\\ &\leq \frac{1}{2}\|u^+\|_{L^\infty(B_R)} + C\left\{\frac{1}{[(1-\theta)R]^{n/p}}\left(\int_{B_R}(u^+)^p\, dx\right)^{\frac{1}{p}} + \|f\|_{L^q(B_R)}\right\}.\end{aligned}$$

Set $h(t) = \|u^+\|_{L^\infty(B_t)}$ for $t \in (0,1]$ so that the previous estimate can be rewritten as

$$h(r) \leq \frac{1}{2}h(R) + \frac{C}{(R-r)^{n/p}}\|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)} \quad \text{for any } 0 < r < R \leq 1.$$

We apply Lemma 3.11 from below to get for any $0 < r < R < 1$

$$h(r) \leq \frac{C}{(R-r)^{n/p}}\|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

Let $R \longrightarrow 1^-$. Hence, for any $0 < \theta < 1$ we get the desired estimate

$$\|u^+\|_{L^\infty(B_\theta)} \leq \frac{C}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

□

At the end of the proof, recall that we invoked the following lemma whose proof can be found in [13].

Lemma 3.11. *Let $h(t) \geq 0$ be bounded in $[\tau_0, \tau_1]$ with $\tau_0 \geq 0$. Suppose for $\tau_0 \leq t < s \leq \tau_1$ we have*

$$h(t) \leq \theta h(s) + \frac{A}{(s-t)^\alpha} + B$$

for some $\theta \in [0,1)$. Then for any $\tau_0 \leq t < s \leq \tau_1$ there holds

$$h(t) \leq c(\alpha, \theta) \left\{ \frac{A}{(s-t)^\alpha} + B \right\}.$$

Moser's iteration can again be applied to prove a closely related high integrability result. We omit its proof but refer the reader to [13] for the details.

Theorem 3.25 (high integrability)**.** *Suppose $a^{ij} \in L^\infty(B_1)$ and $c \in L^{\frac{n}{2}}(B_1)$ satisfy the following assumption:*

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for any } x \in B_1, \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense:

$$\int_{B_1} a^{ij}(x) D_i u \, D_j \varphi + c(x) u \varphi \, dx \leq \int_{B_1} f \varphi \, dx$$

for any non-negative $\varphi \in H_0^1(B_1)$. If $f \in L^q(B_1)$ for some $q \in [\frac{2n}{n+2}, \frac{n}{2})$, then $u^+ \in L^{q^*}_{loc}(B_1)$ for $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$. Moreover, there holds

$$\|u^+\|_{L^{q^*}(B_{1/2})} \leq C \Big\{ \|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)} \Big\}$$

where $C = C(n, \lambda, \Lambda, q, \epsilon(K))$ is a positive constant with

$$\epsilon(K) = \left(\int_{\{|c|>K\}} |c|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

3.7.4 Hölder Regularity: De Giorgi's Approach

For simplicity, we establish the Hölder continuity of weak solutions to homogeneous equations without lower-order terms,

$$Lu \equiv -\sum_{i,j=1}^n D_i \left(a^{ij}(x) D_j u \right) \text{ in } B_1(0) \subset \mathbb{R}^n,$$

where $a^{ij} \in L^\infty(B_1)$ satisfies

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in B_1(0) \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ.

Definition 3.9. *The function $u \in H^1_{loc}(B_1)$ is called a* **subsolution** *(resp.* **supersolution***) of the equation $Lu = 0$ if*

$$\int_{B_1} a^{ij}(x) D_i u \, D_j \varphi \, dx \leq 0 \text{ (resp. } \geq 0) \text{ for all non-negative } \varphi \in H^1_0(B_1).$$

First, we will need the following, which indicates that monotone convex mappings preserve subsolutions and supersolutions. The proof follows from a direct computation and we omit the details (cf. [13] for the proof).

Lemma 3.12. *Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ be convex. Then*

(i) If u is a subsolution and $\Phi' \geq 0$, then $v = \Phi(u)$ is also a subsolution provided that $v \in H^1_{loc}(B_1)$.

(ii) If u is a supersolution and $\Phi' \leq 0$, then $v = \Phi(u)$ is a subsolution provided that $v \in H^1_{loc}(B_1)$.

Next is a Poincaré type inequality. But unlike the more common Poincaré inequalities that assume u belongs to $H^1_0(B_1)$ or an inequality that involves the difference between u and its average, this version says that if $u \in H^1(B_1)$ vanishes in a measurable portion of the domain, then it can be controlled by its gradient in L^2.

Lemma 3.13 (Poincaré–Sobolev)**.** *For any $\epsilon > 0$ there exists a constant $C = C(\epsilon, n)$ such that for $u \in H^1(B_1)$ with $\mu(\{x \in B_1 \mid u = 0\}) \geq \epsilon\mu(B_1)$, there holds*

$$\int_{B_1} u^2 \, dx \leq C \int_{B_1} |Du|^2 \, dx.$$

Proof. Suppose the contrary. Then there is a sequence $\{u_m\} \subset H^1(B_1)$ such that

$$\mu(\{x \in B_1 \mid u = 0\}) \geq \epsilon\mu(B_1), \int_{B_1} u_m^2 \, dx = 1, \int_{B_1} |Du_m|^2 \, dx \longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Hence, we may assume $u_m \longrightarrow u_0 \in H^1(B_1)$ strongly in $L^2(B_1)$ and weakly in $H^1(B_1)$. Clearly, u_0 is a non-zero constant. Thus,

$$\begin{aligned} 0 = \lim_{m \longrightarrow \infty} \int_{B_1} |u_m - u_0|^2 \, dx &\geq \lim_{m \longrightarrow \infty} \int_{\{u_m = 0\}} |u_m - u_0|^2 \, dx \\ &\geq |u_0|^2 \inf_m \mu(\{u_m = 0\}) > 0, \end{aligned}$$

which is a contradiction. □

If u is some positive weak solution, or more generally a supersolution, and it is bounded uniformly away from zero in a measurable portion of the domain, then we can use the previous two lemmas to prove that u is locally bounded away from zero.

Theorem 3.26 (Density)**.** *Suppose u is a positive supersolution in B_2 with*

$$\mu(\{x \in B_1 \,|\, u \geq 1\}) \geq \epsilon \mu(B_1).$$

Then there exists a constant C depending only on ϵ, n, and Λ/λ such that

$$\inf_{B_{1/2}} u \geq C.$$

Proof. We may assume that $u \geq \delta > 0$. Then let $\delta \longrightarrow 0$. By Lemma 3.12, $v = (\log u)^-$ is a subsolution, bounded by $\log \delta^{-1}$. Then Theorem 3.24 implies

$$\sup_{B_{1/2}} v \leq C \left(\int_{B_1} |v|^2 \, dx \right)^{\frac{1}{2}}.$$

Observe that $\mu(\{x \in B_1 \,|\, v = 0\}) = \mu(\{x \in B_1 \,|\, u \geq 1\}) \geq \epsilon \mu(B_1)$. Lemma 3.13 implies

$$\sup_{B_{1/2}} v \leq C \left(\int_{B_1} |Dv|^2 \, dx \right)^{\frac{1}{2}}. \tag{3.89}$$

Set $\varphi = \zeta/u$ for $\zeta \in C_0^1(B_2)$ as the test function. Then

$$0 \leq \int a^{ij}(x) D_i u \, D_j \left(\frac{\zeta^2}{u} \right) dx = -\int \zeta^2 \frac{a^{ij}(x) D_i u \, D_j u}{u^2} \, dx + 2 \int \frac{\zeta a^{ij}(x) D_i u \, D_j \zeta}{u} \, dx,$$

which implies

$$\int \zeta^2 |D \log u|^2 \, dx \leq C \int |D\zeta|^2 \, dx.$$

Thus, for fixed $\zeta \in C_0^1(B_2)$ with $\zeta \equiv 1$ in B_1, we obtain

$$\int_{B_1} |D \log u|^2 \, dx \leq C.$$

Combining this with (3.89) yields

$$\sup_{B_{1/2}} v = \sup_{B_{1/2}} (\log u)^- \leq C,$$

which implies

$$\inf_{B_{1/2}} u \geq e^{-C} > 0.$$

□

The preceding density theorem will be used to control the oscillation of a weak solution u, which is the key ingredient in deriving its local Hölder continuity.

Theorem 3.27 (Oscillation)**.** *Suppose that u is a bounded solution of $Lu = 0$ in B_2. Then there exists a $\gamma = \gamma(n, \Lambda/\lambda) \in (0, 1)$ such that*

$$osc_{B_{1/2}} u \leq \gamma \, osc_{B_1} u. \tag{3.90}$$

Remark 3.13. *The oscillation of f over the set S is given by*

$$osc_S(f) := \sup_{x \in S} f(x) - \inf_{x \in S} f(x).$$

Proof. In fact, local boundedness follows from Theorem 3.24. Set

$$\alpha_1 = \sup_{B_1} u \ \text{ and } \ \beta_1 = \inf_{B_1} u.$$

Consider the solution

$$\frac{u - \beta_1}{\alpha_1 - \beta_1} \ \text{ or } \ \frac{\alpha_1 - u}{\alpha_1 - \beta_1}.$$

Note the following equivalence:

$$u \geq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{2},$$
$$u \leq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{\alpha_1 - u}{\alpha_1 - \beta_1} \geq \frac{1}{2}.$$

Case 1: Suppose that

$$\mu\left(\left\{x \in B_1 \, : \, \frac{2(u - \beta_1)}{\alpha_1 - \beta_1} \geq 1\right\}\right) \geq \frac{1}{2}\mu(B_1).$$

Applying the density theorem to $\frac{u-\beta_1}{\alpha_1-\beta_1} \geq 0$ in B_1, we get for some constant $C > 1$

$$\inf_{B_{1/2}} \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{C},$$

which implies

$$\inf_{B_{1/2}} u \geq \beta_1 + \frac{1}{C}(\alpha_1 - \beta_1).$$

Case 2: Suppose that

$$\mu\left(\left\{x \in B_1 \, : \, \frac{2(\alpha_1 - u)}{\alpha_1 - \beta_1} \geq 1\right\}\right) \geq \frac{1}{2}\mu(B_1).$$

Applying the density theorem as before and noting that $\sup_{B_{1/2}} u = \inf_{B_{1/2}} -u$, we obtain

$$\sup_{B_{1/2}} u \leq \alpha_1 - \frac{1}{C}(\alpha_1 - \beta_1).$$

Now set

$$\alpha_2 = \sup_{B_{1/2}} u \ \text{ and } \ \beta_2 = \inf_{B_{1/2}} u,$$

and note that $\beta_2 \geq \beta_1$ and $\alpha_2 \leq \alpha_1$. In both cases, we have

$$\alpha_2 - \beta_2 \leq \left(1 - \frac{1}{C}\right)(\alpha_1 - \beta_1).$$

This is precisely the estimate (3.90) with $\gamma = 1 - 1/C \in (0,1)$. $\square$

At last, we are now equipped to state and prove De Giorgi's Hölder regularity theorem.

Theorem 3.28 (De Giorgi)**.** *Suppose $Lu = 0$ weakly in B_1. Then there holds*

$$\sup_{B_{1/2}} |u(x)| + \sup_{x,y \in B_{1/2}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C(n, \Lambda/\lambda)\|u\|_{L^2(B_1)},$$

where $\alpha = \alpha(n, \Lambda/\lambda) \in (0,1)$.

Proof. The first part of the estimate follows from Theorem 3.24; that is,

$$\sup_{B_{1/2}} |u(x)| \leq C(n, \Lambda/\lambda)\|u\|_{L^2(B_1)}.$$

We prove the second part of the estimate. Fix any two distinct points $x, y \in B_{1/2}$, set $r = |x-y|$ and let

$$\omega(r) := osc_{B_r}(u) = \sup_{B_r} u - \inf_{B_r} u.$$

By Theorem 3.27 and rescaling, we obtain that

$$\omega(r/2) \leq \gamma\omega(r).$$

Hence, Lemma 3.10 implies that

$$\omega(r) \leq C r^\alpha \omega(1/2) \ \text{ for all } 0 < r \leq 1/2,$$

where $\alpha = \alpha(n, \Lambda/\lambda)$ is some number in $(0,1)$. By Theorem 3.24, we have that

$$\omega(1/2) \leq \sup_{B_{1/2}} |u(x)| \leq C\|u\|_{L^2(B_1)}.$$

Inserting this into the previous estimate yields

$$\omega(r) \leq C r^\alpha \|u\|_{L^2(B_1)},$$

which further implies

$$\sup_{x,y \in B_{1/2}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C(n, \Lambda/\lambda)\|u\|_{L^2(B_1)}.$$

This completes the proof.

$\square$

3.7.5 Hölder Regularity: the Weak Harnack Inequality

We now state and prove the weak Harnack inequality. As a result, we derive Moser's Harnack inequality as a special case, and we combine it with our previous local boundedness result to give another proof of the interior Hölder continuity of weak solutions. Then, we also examine applications of the weak Harnack inequality to obtain a Liouville type theorem and a version of the strong maximum principles for weak solutions.

For simplicity, we only consider elliptic equations without lower order terms. Suppose $U \subset \mathbb{R}^n$, $a^{ij} \in L^\infty(U)$ satisfies

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in U \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants λ and Λ.

Theorem 3.29 (Weak Harnack inequality)**.** *Let $u \in H^1(U)$ be a non-negative supersolution in U, i.e.,*

$$\int_U a^{ij}(x)D_iuD_j\varphi\,dx \geq \int_U f\varphi\,dx \text{ for any non-negative } \varphi \in H_0^1(U). \tag{3.91}$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then for any $B_R \subset U$, there holds for any $p \in (0, \frac{n}{n-2})$ and any $0 < \theta < \tau < 1$,

$$\inf_{B_{\theta R}} u + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)} \geq C\Big(\frac{1}{R^n}\int_{B_{\tau R}} u^p\Big)^{\frac{1}{p}}$$

where $C = C(n, \lambda, \Lambda, p, q, \theta, \tau)$ is a positive constant.

The proof of the weak Harnack inequality and the result on the Hölder continuity of weak solutions will make use of the following result, which is a special case of the local boundedness result of Theorem 3.24.

Theorem 3.30 (local boundedness)**.** *Let $u \in H^1(U)$ be a non-negative subsolution in U in the following sense:*

$$\int_U a^{ij}(x)D_iuD_j\varphi\,dx \leq \int_U f\varphi\,dx \text{ for any non-negative } \varphi \in H_0^1(U).$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then there holds for any $B_R \subset U$, any $r \in (0, R)$, and any $p > 0$,

$$\sup_{B_r} u \leq C\Big\{\frac{1}{(R-r)^{n/p}}\|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\Big\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

Proof of the weak Harnack inequality. We prove this for $R=1$.

Step 1: We prove the result for some $p=p_0>0$. Set $\bar{u}=u+k>0$ for some $k>0$ to be determined below and $v=\bar{u}^{-1}$. First, we derive the equation for $v(x)$. For any non-negative $\varphi\in H^1_0(B_1)$, let the function $\bar{u}^{-2}\varphi$ be the test function in equation (3.91). Then

$$\int_{B_1} a^{ij}(x)D_iu\frac{D_j\varphi}{\bar{u}^2}\,dx-2\int_{B_1}a^{ij}(x)D_iuD_j\bar{u}\frac{\varphi}{\bar{u}^3}\,dx\ge\int_{B_1}f\frac{\varphi}{\bar{u}^2}\,dx$$

Note that $D\bar{u}=Du$ and $Dv=-\bar{u}^2D\bar{u}$. Therefore, we obtain

$$\int_{B_1}a^{ij}(x)D_jvD_i\varphi+\bar{f}v\varphi\,dx\le 0\;\text{ where }\;\bar{f}:=\frac{f}{\bar{u}}.$$

That is, v is a non-negative subsolution to some homogeneous equation. Choose $k=\|f\|_{L^q(U)}$ if $f\not\equiv 0$. Otherwise, choose arbitrary $k>0$ and let $k\to 0$. Note $\|\bar{f}\|_{L^q(B_1)}\le 1$. Thus, Theorem 3.30 implies that for any $\tau\in(\theta,1)$ and any $p>0$,

$$\sup_{B_\theta}\bar{u}^{-p}\le C\int_{B_\tau}\bar{u}^{-p}\,dx,$$

that is, we deduce the desired estimate

$$\inf_{B_\theta}\bar{u}\ge C\Big(\int_{B_\tau}\bar{u}^{-p}\,dx\Big)^{-\frac{1}{p}}=C\Big(\int_{B_\tau}\bar{u}^{-p}\,dx\int_{B_\tau}\bar{u}^{p}\,dx\Big)^{-\frac{1}{p}}\Big(\int_{B_\tau}\bar{u}^{p}\,dx\Big)^{\frac{1}{p}},$$

where $C=C(n,\lambda,\Lambda,p,q,\theta,\tau)$ is a positive constant. The main step here is to prove there exists a $p_0>0$ such that

$$\int_{B_\tau}\bar{u}^{-p_0}\,dx\cdot\int_{B_\tau}\bar{u}^{p_0}\,dx\le C(n,\lambda,\Lambda,p,q,\tau).\tag{3.92}$$

To show this, it suffices to prove the following claim:
For any $\tau<1$,

$$\int_{B_\tau}e^{p_0|w|}\,dx\le C(n,\lambda,\Lambda,p,q)\tau^n\text{ or }C(n,\lambda,\Lambda,p,q,\tau)\tag{3.93}$$

where

$$w=\log\bar{u}-\beta\;\text{ with }\;\beta=|B_\tau|^{-1}\int_{B_\tau}\log\bar{u}\,dx,$$

since this claim and the fact that $-p_0|w|\le\pm p_0w\le p_0|w|$ would imply that

$$\begin{aligned}\int_{B_r}\bar{u}^{-p_0}\,dx(\int_{B_r}\bar{u}^{p_0}\,dx)&=\int_{B_r}e^{-p_0\beta}e^{\log\bar{u}^{p_0}}\,dx\int_{B_r}e^{p_0\beta}e^{\log\bar{u}^{-p_0}}\,dx\\&=\int_{B_r}e^{-p_0w}\,dx\int_{B_r}e^{p_0w}\,dx\le C(n,\lambda,\Lambda,p,q,\tau).\end{aligned}$$

To prove estimate (3.93), we notice that it follows directly from the John-Nirenberg lemma, i.e., Lemma 3.2, provided that we show $w \in BMO$, i.e.,

$$\frac{1}{r^n}\int_{B_r}|w-w_{y,r}|\,dx \leq C.$$

We first derive the equation for w. As before, consider $\bar{u}^{-1}\varphi$ to be the test function in (3.91) and assume that φ is non-negative with $\varphi \in L^\infty(B_1)\cap H_0^1(B_1)$. By direct calculations and the fact that $Dw = \bar{u}^{-1}Du$, we get that

$$\int_{B_1} a^{ij}(x)D_iwD_j(w\varphi)\,dx \leq \int_{B_1} a^{ij}(x)D_iwD_j\varphi\,dx + \int_{B_1} -\bar{f}\varphi\,dx \tag{3.94}$$

for any non-negative $\varphi \in L^\infty(B_1)\cap H_0^1(B_1)$. Replace φ by φ^2 in (3.94). Then Hölder's inequality yields

$$\int_{B_1}|Dw|^2\varphi^2\,dx \leq C\Big(\int_{B_1}|D\varphi|^2\,dx + \int_{B_1}|\bar{f}|\varphi^2\,dx\Big). \tag{3.95}$$

Furthermore, Hölder's inequality and the Sobolev embedding imply

$$\int_{B_1}|\bar{f}|\varphi^2\,dx \leq \|\bar{f}\|_{L^{n/2}(B_1)}\|\varphi\|^2_{L^{\frac{2n}{n-2}}(B_1)} \leq C(n,q)\|D\varphi\|^2_{L^2(B_1)}.$$

Hence,

$$\int_{B_1}|Dw|^2\varphi^2\,dx \leq C(n,q,\lambda,\Lambda)\int_{B_1}|D\varphi|^2\,dx. \tag{3.96}$$

Here, we can choose φ to be in $C_0^1(B_1)$. Moreover, for any $B_{2r}(y)\subset B_1$, we can choose φ with $supp\,\varphi \subset B_{2r}(y)$, $\varphi \equiv 1$ in $B_r(y)$, and $|D\varphi| \leq \frac{2}{r}$. Then

$$\int_{B_r(y)}|Dw|^2\,dx \leq Cr^{n-2}.$$

Hence, Poincare's inequality yields

$$\frac{1}{r^n}\int_{B_r(y)}|w-w_{y,r}|\,dx \leq \frac{1}{r^{n/2}}\Big(\int_{B_r(y)}|w-w_{y,r}|^2\,dx\Big)^{\frac{1}{2}} \leq \frac{1}{r^{n/2}}\Big(r^2\int_{B_r(y)}|Dw|^2\,dx\Big)^{\frac{1}{2}} \leq C.$$

That is, $w \in BMO$ and this proves the claim.

Step 2: We now verify the result for any $p \in (0, \frac{n}{n-2})$, but we only sketch the main steps as it is similar to the proof of Theorem 3.24. It suffices to prove the following claim. Namely, by the existence of p_0 from Step 1, Moser's iteration scheme yields, for any $0 < r_1 < r_2 < 1$ and $0 < p_2 < p_1 < \frac{n}{n-2}$,

$$\Big(\int_{B_{r_1}}\bar{u}^{p_1}\,dx\Big)^{\frac{1}{p_1}} \leq C(n,q,\lambda,\Lambda,r_1,r_2,p_1,p_2)\Big(\int_{B_{r_2}}\bar{u}^{p_2}\,dx\Big)^{\frac{1}{p_2}}. \tag{3.97}$$

To start, we take $\varphi = \bar{u}^{-\beta-1}\eta^2$ for $\beta \in (0,1)$ as the test function in (3.91). Then, we can establish that

$$\int_{B_1} |D\bar{u}|^2 \bar{u}^{-\beta-1}\eta^2\,dx \le C\Big\{\frac{1}{\beta^2}\int_{B_1} |D\eta|^2 \bar{u}^{1-\beta}\,dx + \frac{1}{\beta}\int_{B_1} \frac{|f|}{k}\eta^2 \bar{u}^{1-\beta}\,dx\Big\}.$$

Set $\gamma = 1-\beta \in (0,1)$ and $w = \bar{u}^{\gamma/2}$. Then we have

$$\int |Dw|^2\eta^2\,dx \le \frac{C}{(1-\gamma)^\alpha}\int w^2(|D\eta|^2+\eta^2)\,dx$$

or

$$\int |D(w\eta)|^2 dx \le \frac{C}{(1-\gamma)^\alpha}\int w^2(|D\eta|^2+\eta^2)\,dx$$

for some positive $\alpha > 0$. By the Sobolev embedding and a proper choice of a cutoff function with $\chi = n/(n-2)$, we obtain for any $\gamma \in (0,1)$ and $0 < r < R < 1$,

$$\Big(\int_{B_r} w^{2\chi}\,dx\Big)^{1/\chi} \le \frac{C}{(1-\gamma)^\alpha}\frac{1}{(R-r)^2}\int_{B_R} w^2\,dx,$$

or

$$\begin{aligned}\Big(\int_{B_r} \bar{u}^{\gamma\chi}\,dx\Big)^{1/\gamma\chi} &\le \Big(\frac{C}{(1-\gamma)^\alpha}\frac{1}{(R-r)^2}\Big)^{1/\gamma}\Big(\int_{B_R} \bar{u}^{\gamma}\,dx\Big)^{1/\gamma}\\ &\le \Big(\frac{C(1+\gamma)^{1+\sigma}}{R-r}\Big)^{2/\gamma}\Big(\int_{B_R} \bar{u}^{\gamma}\,dx\Big)^{1/\gamma}\end{aligned} \tag{3.98}$$

for some $\sigma > 0$. We may iterate this last estimate finitely-many times to get (3.97). $\square$

A special case of the weak Harnack inequality is Moser's version.

Theorem 3.31 (Moser's Harnack inequality)**.** *Let $u \in H^1(U)$ be a non-negative solution in U, i.e.,*

$$\int_U a^{ij}(x)D_i u D_j\varphi\,dx = \int_U f\varphi\,dx \;\; \text{for any} \;\; \varphi \in H_0^1(U).$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then there holds for any $B_R \subset U$,

$$\max_{B_{R/2}} u \le C\Big(\min_{B_{R/2}} u + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\Big)$$

where $C = C(n,\lambda,\Lambda,q)$ is a positive constant.

The proof of Moser's version of the Harnack inequality follows from the weak version and Lemma 3.9.

Proof of Moser's Harnack Inequality. Define $\Phi(p,r)$ by

$$\Phi(p,r) := \Big(\int_{B_r} \bar{u}^p \, dx \Big)^{1/p}.$$

Then (3.98) implies the estimate

$$\Phi(\chi\gamma, r) \leq \Big(\frac{C(1+\gamma)^{\sigma+1}}{R-r} \Big)^{2/\gamma} \Phi(\gamma, R). \tag{3.99}$$

Set for $m = 0, 1, 2, 3, \ldots,$

$$\gamma = \gamma_m = \chi^m p \ \text{ and } \ r_m = 1/2 + 2^{-(m+1)}.$$

Then, by iterating estimate (3.99), we get

$$\Phi(\chi^m \gamma, 1/2) \leq (C\chi)^{2(1+\sigma)\sum m\chi^{-m}} \Phi(p, 1).$$

By sending $m \longrightarrow \infty$ here and applying Lemma 3.9, we arrive at

$$\sup_{B_{1/2}} \bar{u} \leq C\Phi(p,1).$$

The desired estimate follows from this and the weak Harnack inequality. □

Now, our goal is to establish the Hölder continuity of weak solutions using the local boundedness result and Moser's Harnack inequality.

Corollary 3.3 (Hölder continuity)**.** *Let $u \in H^1(U)$ be a solution of the equation in U:*

$$\int_U a^{ij}(x) D_i u D_j \varphi \, dx = \int_U f\varphi \, dx \ \textit{ for any } \ \varphi \in H_0^1(U).$$

Suppose $f \in L^q(U)$ for some $q > n/2$. Then $u \in C^\alpha(U)$ for some $\alpha \in (0,1)$ depending only on n, q, λ and Λ. Moreover, there holds for any $B_R \subset U$

$$|u(x) - u(y)| \leq C\Big(\frac{|x-y|}{R} \Big)^\alpha \Big\{ \Big(\frac{1}{R^n} \int_{B_R} u^2 \, dx \Big)^{\frac{1}{2}} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \Big\}$$

for any $x, y \in B_{R/2}$ where $C = C(n, \lambda, \Lambda, q)$ is a positive constant.

Proof. We prove the estimate for the case $R = 1$. Set for $r \in (0,1)$

$$M(r) = \max_{B_r} u \ \text{ and } \ m(r) = \min_{B_r} u.$$

Then $M(r) < \infty$ and $m(r) > -\infty$. It suffices to prove for any $r < 1/2$,

$$\omega(r) := M(r) - m(r) \leq Cr^\alpha \Big\{ \Big(\int_{B_1} u^2 \, dx \Big)^{\frac{1}{2}} + \|f\|_{L^q(B_1)} \Big\}. \tag{3.100}$$

Set $\delta = 2 - n/q$ and apply Theorem 3.31 to $M(r) - u \geq 0$ in B_r to get

$$\sup_{B_{r/2}}(M(r) - u) \leq C\Big\{\inf_{B_{r/2}}(M(r) - u) + r^\delta \|f\|_{L^q(B_r)}\Big\}.$$

Combining this with the definitions of the supremum and infimum, we get

$$\begin{aligned}\inf_{B_{r/2}}(M(r) - u) \leq & \sup_{B_{r/2}}(M(r) - u) \\ & \leq C\Big\{\inf_{B_{r/2}}(M(r) - u) + r^\delta \|f\|_{L^q(B_r)}\Big\} \leq C\Big\{\sup_{B_{r/2}}(M(r) - u) + r^\delta \|f\|_{L^q(B_r)}\Big\}.\end{aligned}$$

Hence,

$$M(r) - m(r/2) \leq C\Big\{(M(r) - M(r/2)) + r^\delta \|f\|_{L^q(B_r)}\Big\}. \tag{3.101}$$

Likewise, applying the same argument to $u - m(r) \geq 0$ in B_r, we get

$$M(r/2) - m(r) \leq C\Big\{(m(r/2) - m(r)) + r^\delta \|f\|_{L^q(B_r)}\Big\}. \tag{3.102}$$

Adding (3.101) and (3.102) together yields

$$\omega(r) + \omega(r/2) \leq C\Big\{(\omega(r) - \omega(r/2)) + r^\delta \|f\|_{L^q(B_r)}\Big\}$$

or

$$\omega(r/2) \leq \gamma\omega(r) + Cr^\delta \|f\|_{L^q(B_r)}$$

for some $\gamma = (C-1)/(C+1) < 1$.

Apply Lemma 3.10 with μ is chosen such that $\alpha = (1-\mu)\log\gamma/\log\tau < \mu\delta$. Then

$$\omega(\rho) \leq C\rho^\alpha\{\omega(1/2) + \|f\|_{L^q(B_1)}\} \text{ for any } \rho \in (0, 1/2]. \tag{3.103}$$

On the other hand, Theorem 3.30 implies

$$\omega(1/2) \leq C\Big\{\Big(\int_{B_1} u^2\, dx\Big)^{\frac{1}{2}} + \|f\|_{L^q(B_1)}\Big\}$$

and inserting this into (3.103) completes the proof of the corollary. □

3.7.6 Further Applications of the Weak Harnack Inequality

A Liouville theorem

First, we point out an application of Lemma 3.10. Namely, we can derive the following Liouville theorem.

Theorem 3.32. *Suppose $u \in H^1(U)$ is a solution to the homogeneous equation in $\mathbb{R}^n$:*

$$\int_{\mathbb{R}^n} a^{ij}(x) D_i u D_j \varphi \, dx = 0 \ \text{ for any } \ \varphi \in H_0^1(\mathbb{R}^n).$$

If u is bounded, then u is constant.

Proof. From the previous corollary, we showed that there exists a $\gamma < 1$ such that

$$\omega(r) \leq \gamma \omega(2r).$$

By iteration, we obtain

$$\omega(r) \leq \gamma^k \omega(2^k r) \to 0 \ \text{ as } \ k \to \infty$$

since $\omega(2^k r) \leq C$ if u is bounded. Hence, for any $r > 0$,

$$\omega(r) = 0.$$

Thus, $u \equiv$ *constant*. □

Maximum principles for weak solutions

An application of the weak Harnack inequality is the strong maximum principle adapted for weak solutions. However, we introduce some necessary definitions and consider the weak maximum principle for weak solutions. We say that $u \in H^1(U)$ satisfies $u \leq 0$ on ∂U if its positive part $u^+ = \max\{u, 0\}$ belongs to $H_0^1(U)$. Of course if u is continuous in a neighborhood of ∂U then u satisfies $u \leq 0$ on ∂U if the inequality holds in the classical pointwise sense. Likewise, we say $u \geq 0$ on ∂U if $-u \leq 0$ on ∂U; and $u \leq v \in H^1(U)$ on ∂U if $u - v \leq 0$ on ∂U. As usual, we take

$$Lu = -D_i(a^{ij}(x) D_j u)$$

and solutions, supersolutions, and subsolutions associated with this elliptic operator are understood in the distributional sense.

Theorem 3.33 (Weak Maximum Principle for Weak Solutions)**.** *Let $u \in H^1(U)$.*

(a) If $Lu \leq 0$ in U, then $\sup_U u \leq \sup_{\partial U} u^+$.

(b) If $Lu \geq 0$ in U, then $\inf_U u \geq \inf_{\partial U} u^-$.

Proof. Since $Lu \leq 0$ in U in the distribution sense, we write

$$\int_U a^{ij}(x) D_j u D_i v \, dx \leq 0$$

for all non-negative $v \in H_0^1(U)$. If we set $\ell = \sup_{\partial U} u^+$ and take $v = \max\{u - \ell, 0\}$, then $v \in H_0^1(U)$, $Dv = Du$ if $u - \ell > 0$ and $Dv = 0$ if $u - \ell \leq 0$. We proceed by contradiction.

That is, assume $v > 0$ or $u > \ell$ in some subset $B \subset\subset U$ with $\mu(B) > 0$; otherwise, if $v \equiv 0$ then we would be done. Clearly, $Dv = Du$ within B; but the positivity of $(a^{ij}(x))$ and the uniform ellipticity condition imply that

$$\int_B |Dv|^2 \, dx \leq 0,$$

and we get that v, and therefore u, is constant in a subset of U with positive measure. At the same time, a basic result guarantees $Du = 0$ *a.e.* in this subset and we deduce a contradiction. This completes the proof for part (a). Part (b) follows along a similar argument; namely, we can apply the previous proof to $-Lu \leq 0$ and the fact that $\inf_D u = -\sup_D(-u)$.

□

From this, we immediately deduce a uniqueness result.

Corollary 3.4. *Let $u \in H_0^1(U)$ satisfy $Lu = 0$ in U. Then $u = 0$ in U.*

We are now ready to introduce the strong version of the maximum principle adapted for weak solutions. Unlike the weak maximum principle above, we are only assuming the weak solution belongs to $H^1(U)$. We do not assume the solution vanishes at the boundary in the trace sense, i.e., it does not necessarily belong to $H_0^1(U)$. The Harnack inequality plays an essential role in its proof.

Theorem 3.34 (Strong Maximum Principle for Weak Solutions)**.** *Let U be a bounded and open subset and let $u \in H^1(U)$ satisfy $Lu \leq 0$ in U. Then, if for some ball $B \subset\subset U$ we have*

$$\sup_B u = \sup_U u \geq 0, \tag{3.104}$$

the function u must be constant in U.

Proof. Denote $B = B_R(y)$ and without loss of generality, we can assume that $B_{4R}(y) \subset U$. Now let $M = \sup_U u$ and then apply the weak Harnack inequality (see Theorem 3.29) with $p = 1$ to the supersolution $v = M - u$. Namely, we use the following dilated version of the weak Harnack inequality with $p = 1$:

$$R^{-n} \|v\|_{L^1(B_{2R}(y))} \leq C \inf_{B_R(y)} v.$$

Hence,

$$R^{-n} \int_{B_{2R}} (M - u) \, dx \leq C \inf_B (M - u) = 0$$

and so $u \equiv M$ in B_{2R}. Therefore, supremum of u is attained for a larger ball in U. We can then show $u \equiv M$ in U by a simple covering argument. □

Remark 3.14. *Likewise, we have an analogous result which states the solution to $Lu \geq 0$ in U is constant whenever it attains an interior minimum.*

CHAPTER 4

Viscosity Solutions and Fully Nonlinear Equations

4.1 Introduction

This chapter introduces a very weak concept of solution for second-order elliptic equations called viscosity solutions. To simplify our presentation, the results given here are for equations involving linear elliptic operators without lower order terms, but they can certainly be extended to fully nonlinear elliptic equations of the type

$$F(D^2u, u, x) = f(x) \text{ in } U,$$

where $F : \mathbb{R}^{n\times n} \times \mathbb{R} \times \mathbb{R}^n$ is usually a monotone and convex mapping possibly nonlinear in D^2u and u. For a nice introductory treatment of this topic, we refer the reader to Caffarelli and Cabré [3].

The advantage of considering the notion of viscosity solution is it allows us to consider elliptic equations in non-divergence form, and it extends the notion of classical solutions. Another advantage is that viscosity solutions are stable under local uniform convergence in both u and F and because existence and uniqueness results for such solutions can be obtained under far more general conditions. In fact, in the definition given below, notice that we can make sense of such solutions without resorting to differentiating the equations directly. This was a major obstacle in extending elliptic theory to equations having non-divergence form, since the usual procedure of integrating by parts and treating equations in the distribution sense was not generally possible, or the usual notions of solution was not always guaranteed to exist in this context. Thus, finding a successful framework that circumvents this obstacle was a tremendous breakthrough in the modern theory of elliptic partial differential equations.

The results we establish below should be reminiscent of those for elliptic equations in divergence form studied earlier, however, we obtain the results via perturbation methods relying heavily on approximation and density arguments. More precisely, we shall give a concise introduction, develop the Alexandroff maximum principle along with a Harnack inequality for viscosity solutions. Then we use these to develop the interior Schauder and $W^{2,p}$ regularity estimates for viscosity solutions. Global versions of these regularity results without proof are also provided at the end of the chapter.

Let U be a bounded and connected domain in $\mathbb{R}^n$ and (a^{ij}) is of class $C(U)$ and satisfies

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for any $x \in U$ and any $\xi \in \mathbb{R}^n$. We consider the operator L in U defined by

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u \text{ for } u \in C^2(U). \tag{4.1}$$

Throughout, we shall assume that f belongs to $C(U)$.

Definition 4.1. *The function $u \in C(U)$ is said to be a* **viscosity supersolution** *(respectively* **viscosity subsolution***) of the equation*

$$Lu = f \text{ in } U \tag{4.2}$$

if for any $x_0 \in U$ and any function $\varphi \in C^2(U)$ such that $u - \varphi$ has a local minimum (respectively, local maximum) at x_0 there holds

$$L\varphi(x_0) \geq f(x_0) \ (\textit{respectively}, L\varphi(x_0) \leq f(x_0)).$$

The following definition of solution should be compared with the result of Theorem 1.10.

Definition 4.2. *We say $u \in C(U)$ is a* **viscosity solution** *of equation* (4.1) *if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 4.1. *By density, the C^2 function φ in the above definitions may be replaced by quadratic polynomials.*

Next we look at the class of all solutions to all elliptic equations. First we make the following important observation. Let $e_1, e_2, \ldots, e_n$ be the eigenvalues of the Hessian matrix $D^2\varphi(x_0)$ where φ is any C^2 function at $x_0 \in U$. We have the following chain of equivalent estimates:

$$\begin{aligned}
\sum_{i,j=1}^{n} a^{ij}(x_0)D_{ij}\varphi(x_0) \leq 0 &\Longleftrightarrow \sum_{i=1}^{n} \alpha_i e_i \leq 0 \text{ for } \alpha_i \in [\lambda, \Lambda], \\
&\Longleftrightarrow \sum_{e_i>0} \alpha_i e_i + \sum_{e_i<0} \alpha_i e_i \leq 0, \\
&\Longleftrightarrow \sum_{e_i>0} \alpha_i e_i \leq \sum_{e_i<0} \alpha_i(-e_i),
\end{aligned}$$

where the last line implies

$$\lambda \sum_{e_i>0} \alpha_i e_i \le \Lambda \sum_{e_i<0} \alpha_i(-e_i).$$

Namely, if u is a "supersolution," then the positive eigenvalues of the Hessian matrix $D^2\varphi(x_0)$ are controlled by its negative eigenvalues. This motivates the following definition.

Definition 4.3. *Suppose $f \in C(U)$ and λ and Λ are two positive constants. We define $u \in C(U)$ to belong to $\mathcal{S}^+(\lambda, \Lambda, f)$ if for any $x_0 \in U$ and any function $\varphi \in C^2(U)$ such that $u - \varphi$ has a local minimum at x_0, there holds*

$$\lambda \sum_{e_i(x_0)>0} e_i(x_0) + \Lambda \sum_{e_i<0} e_i(x_0) \ge f(x_0),$$

where $e_1(x_0), e_2(x_0), \ldots, e_n(x_0)$ are eigenvalues of the Hessian matrix $D^2\varphi(x_0)$.

Similarly, we define $u \in C(U)$ to belong to $\mathcal{S}^-(\lambda, \Lambda, f)$ if for any $x_0 \in U$ and any function $\varphi \in C^2(U)$ such that $u - \varphi$ has a local maximum at x_0, there holds

$$\Lambda \sum_{e_i(x_0)>0} e_i(x_0) + \lambda \sum_{e_i<0} e_i(x_0) \le f(x_0).$$

We denote $\mathcal{S}(\lambda, \Lambda, f) = \mathcal{S}^+(\lambda, \Lambda, f) \cap \mathcal{S}^-(\lambda, \Lambda, f)$

Notice that any viscosity supersolution of (4.2) belongs to the class $\mathcal{S}^+(\lambda, \Lambda, f)$. In fact, the class $\mathcal{S}^+(\lambda, \Lambda, f)$ and $\mathcal{S}^-(\lambda, \Lambda, f)$ also include solutions to fully nonlinear equations such as the Pucci equations.

We say the matrix $A = (a^{ij})$ belongs to the class $A_{\lambda,\Lambda}$ with any two constants $\lambda, \Lambda > 0$ if A is symmetric and

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for } x \in U,\ \xi \in \mathbb{R}^n$$

so that its eigenvalues belong to $[\lambda, \Lambda]$.

Now, for any symmetric matrix $M = (m^{ij})$, we define the Pucci extremal operators:

$$\mathcal{M}^-(M) = \mathcal{M}^-(\lambda, \Lambda, M) = \inf_{A \in A_{\lambda,\Lambda}} a^{ij}m^{ij},$$
$$\mathcal{M}^+(M) = \mathcal{M}^+(\lambda, \Lambda, M) = \sup_{A \in A_{\lambda,\Lambda}} a^{ij}m^{ij}.$$

Then Pucci's equations are given by

$$\mathcal{M}^-(\lambda, \Lambda, M) = f,$$
$$\mathcal{M}^+(\lambda, \Lambda, M) = g,$$

for some functions $f, g \in C(U)$. Indeed, we can show that

$$\mathcal{M}^-(\lambda, \Lambda, M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$
$$\mathcal{M}^+(\lambda, \Lambda, M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_1, e_2, \dots, e_n$ are eigenvalues of M. Hence, $u \in \mathcal{S}^+(\lambda, \Lambda, f)$ if and only if

$$\mathcal{M}^-(\lambda, \Lambda, D^2 u) \leq f$$

in the viscosity sense, i.e., for any $\varphi \in C^2(U)$ such that $u - \varphi$ has a local minimum at $x_0 \in U$ there holds

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0)) \leq f(x_0).$$

An analogous statement holds for $u \in \mathcal{S}^-(\lambda, \Lambda, f)$ and viscosity subsolutions.

By definition of $\mathcal{M}^-$ and $\mathcal{M}^+$, we can check that for any two symmetric matrices M and N,

$$\begin{aligned} \mathcal{M}^-(M) + \mathcal{M}^-(N) &\leq \mathcal{M}^-(M+N) \leq \mathcal{M}^+(M) + \mathcal{M}^-(N) \\ &\leq \mathcal{M}^+(M+N) \leq \mathcal{M}^+(M) + \mathcal{M}^+(N). \end{aligned}$$

This will be an important property we invoke later in establishing the regularity of viscosity solutions. We now establish the **Alexandroff maximum principle** for viscosity solutions, and we may think of it as a replacement of the energy inequality for weak solutions to elliptic equations in divergence form. The Alexandroff maximum principle is sometimes called the **Alexandroff-Bakelman-Pucci estimate**. First, recall that L defined in $\mathbb{R}^n$ is said to be **affine** if

$$L(x) = \ell_0 + \ell(x),$$

where $\ell_0 \in \mathbb{R}$ and ℓ is a linear function. We denote the **convex envelope** of a function v defined in U by

$$\Gamma(v)(x) = \sup_L \{L(x) \,:\, L \leq v \text{ in } U,\, L \text{ is an affine function}\}$$

for any $x \in U$. The function Γ is indeed a convex function on U, and it is the largest possible affine function below of v. Moreover, the set of points x in which $\Gamma(v)$ touches v from below, i.e., the set $\{v = \Gamma(v)\}$, is called the (lower) contact set of v. The points in the contact set are called contact points. The following lemma is the Alexandroff maximum principle and note that u is not required to be a solution to any elliptic equation. The classical version is stated as follows, which we provide without proof (see Lemma 3.4 in [3] and Section 9.1 in [11] for detailed proofs).

Lemma 4.1. *Suppose u is a $C^{1,1}$ function in B_1 with $u \geq 0$ on ∂B_1. Then*

$$\sup_{B_1} u^- \leq C(n)\Big(\int_{B_1 \cap \{u=\Gamma_u\}} \det(D^2 u)\, dx\Big)^{1/n},$$

where Γ_u is the convex envelope of $-u^- = \min\{u, 0\}$.

The version of this for viscosity solutions is the following, which we will prove with the help of Lemma 4.1.

Theorem 4.1 (Alexandroff Maximum Principle)**.** *Suppose u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in B_1 with $u \geq 0$ on ∂B_1 for some $f \in C(U)$. Then*

$$\sup_{B_1} u^- \leq C(n, \lambda, \Lambda)\Big(\int_{B_1 \cap \{u=\Gamma_u\}} (f^+)^n\, dx\Big)^{1/n},$$

where Γ_u is the convex envelope of $-u^- = \min\{u, 0\}$.

Proof. The goal is to ultimately apply Lemma 4.1 to the convex envelope $\Gamma_u(x)$. Namely, we need to prove that Γ_u belongs to $C^{1,1}(B_1)$ and at a contact point x_0, we have that

$$f(x_0) \geq 0 \tag{4.3}$$

and

$$L(x) \leq \Gamma_u(x) \leq L(x) + C(n, \lambda, \Lambda)(f(x_0) + \epsilon(x))|x - x_0|^2 \tag{4.4}$$

for some affine function L and any x sufficiently close to x_0 with $\epsilon(x) = o(1)$ as $x \longrightarrow x_0$. Once we prove this claim, clearly (4.4) implies that

$$\det(D^2\Gamma_u)(x) \leq C(n, \lambda, \Lambda) f(x)^n \text{ for a.e. } x \in \{u = \Gamma_u\}.$$

So Lemma 4.1 applied to the function Γ_u implies the result. Therefore, it remains to prove the claim.

Let x_0 be a contact point, i.e., $u(x_0) = \Gamma_u(x_0)$. Without loss of generality, assume $x_0 = 0$. We may also assume, after subtracting a supporting plane at $x_0 = 0$ if necessary, that $u \geq 0$ in B_1 with $u(0) = 0$. Take $h(x) = -\epsilon|x|^2/2$ in B_1. Clearly, $u - h$ has a minimum at 0, and note that the eigenvalues of $D^2h(0)$ is just $-\epsilon$ with multiplicity n. By definition of $\mathcal{S}^+(\lambda, \Lambda, f)$, we have that

$$-n\Lambda\epsilon \leq f(0).$$

We obtain (4.3) after sending $\epsilon \longrightarrow 0$ in the preceding estimate.

Finally, to obtain estimate (4.4), we will prove

$$0 \leq \Gamma_u(x) \leq C(n, \lambda, \Lambda)(f(0) + \epsilon(x))|x|^2 \text{ for } x \in B_1,$$

where $\epsilon(x) = o(1)$ as $x \longrightarrow 0$.

We need to get an estimate for

$$C_r = \frac{1}{r^2} \max_{B_r} \Gamma_u$$

for small $r > 0$. By convexity, Γ_u attains its maximum in the closed ball $\bar{B}_r$ at some point on the boundary, say at $(0, \ldots, 0, r)$. Now the set $\{x \in B_1 : \Gamma_u(x) \leq \Gamma_u(0, \ldots, 0, r)\}$ is convex and contains B_r. Hence,

$$\Gamma_u(x', r) \geq \Gamma_u(0, \ldots, 0, r) = C_r r^2 \text{ for any } x = (x', r) \in B_1.$$

Choose a positive number N to be specified at a later time. Set

$$R_r = \{(x', x_n) : |x'| \leq Nr, |x_n| \leq r\}.$$

We construct a quadratic polynomial that touches u from below in R_r and curves upward very steeply. Set, for some $b > 0$,

$$h(x) = (x_n + r)^2 - b|x'|^2.$$

Then,

(a) for $x_n = -r$, $h \leq 0$;

(b) for $|x'| = Nr$, $h \leq (4 - bN^2)r^2 \leq 0$ if we take $b = 4/N^2$;

(c) for $x_n = r$, $h = 4r^2 - b|x'|^2 \leq 4r^2$.

Therefore, if we take

$$\tilde{h}(x) = \frac{C_r}{4} h(x) = \frac{C_r}{4}\Big((x_n + r)^2 - \frac{4}{N^2}|x'|^2\Big),$$

and since Γ_u is the convex envelope of u, we have $\tilde{h} \leq \Gamma_u \leq u$ on ∂R_r. Moreover, $\tilde{h}(0) = C_r r^2/4 > 0 = \Gamma_u(0) = u(0)$. Then, after lowering $\tilde{h}$ if necessary, we deduce that $u - \tilde{h}$ has a local minimum in the interior of R_r. It is easily checked that the eigenvalues of $D^2\tilde{h}$ are

$$C_r/2, -2C_r/N^2, \ldots, -2C_r/N^2.$$

Hence, by definition of $\mathcal{S}^+(\lambda, \Lambda, f)$, we have that

$$\lambda \frac{C_r}{2} - 2\Lambda(n-1)\frac{C_r}{N^2} \leq \max_{R_r} f.$$

We can now choose N suitably large, which depends only on n, λ and Λ, so that

$$2\Lambda(n-1)/N^2 \leq \lambda/4.$$

Thus, we obtain

$$C_r \le \frac{4}{\lambda} \max_{R_r} f; \text{ that is, } \max_{B_r} \Gamma_u \le \frac{4}{\lambda} r^2 \max_{R_r} f.$$

Hence,

$$\Gamma_u(x) \le \max_{B_r} \Gamma_u \le \frac{4}{\lambda} \epsilon(r) r^2,$$

where $\epsilon(|x|) = \epsilon(r) = \max_{R_r} f = o(1)$. This completes the proof.

□

Finally, we end this section with a basic result as a consequence of the Calderon-Zygmund decompostion. We will need this result when establishing the Harnack inequality and the regularity theory for viscosity solutions. Here we work in dyadic cubes rather than balls. Q denotes such a dyadic cube after refinement of a given Euclidean domain. We often use $Q_\ell(x_0)$ to denote a dyadic cube centered at $x_0 \in \mathbb{R}^n$ with side length ℓ. Sometimes we omit x_0 if $x_0 = 0$, i.e., $Q_\ell(0) = Q_\ell$.

Lemma 4.2. *Suppse measurable sets $A \subset B \subset Q_1$ have the following properties.*

(a) $|A| < \delta$ for some $\delta \in (0,1)$;

(b) for any dyadic cube Q, $|A \cap Q| \ge \delta |Q|$ implies $\tilde{Q} \subset B$ for the predecessor $\tilde{Q}$ of Q.

Then $|A| \le \delta |B|$.

4.2 A Harnack Inequality

Theorem 4.2 (Harnack inequality)**.** *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1 with $u \ge 0$ in B_1 for some $f \in C(B_1)$. Then*

$$\sup_{B_{1/2}} u \le C\Big(\inf_{B_{1/2}} u + \|f\|_{L^n(B_1)} \Big) \tag{4.5}$$

where C is a positive constant depending only on n, λ and Λ.

As we have encountered already, Harnack type inequalities imply the interior Hölder regularity of solutions. Thus, we have the following result whose proof we omit but follows similarly to that of Corollary 3.3.

Corollary 4.1. *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1 for some $f \in C(B_1)$. Then $u \in C^\alpha(B_1)$ for some $\alpha \in (0,1)$ depending only on n, λ, and Λ. In particular,*

$$|u(x) - u(y)| \le C|x-y|^\alpha \Big(\sup_{B_1} |u| + \|f\|_{L^n(B_1)} \Big) \ \textit{for any } x, y \in B_{1/2}.$$

The main ingredient in proving the Harnack inequality is the following result.

Proposition 4.1. *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ with $u \geq 0$ in $Q_{4\sqrt{n}}$ for some $f \in C(Q_{4\sqrt{n}})$. Then there exist two positive constants ϵ_0 and C, depending only on n, λ, and Λ, such that if*

$$\inf_{Q_{1/4}} u \leq 1 \ \text{ and } \ \|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0,$$

then

$$\sup_{Q_{1/4}} u \leq C.$$

To see how Theorem 4.2 follows from this, consider the function

$$u_\delta = \frac{u}{\inf_{Q_{1/4}} u + \delta + \epsilon_0^{-1}\|f\|_{L^n(Q_{4\sqrt{n}})}} \ (\delta > 0),$$

provided that $u \in \mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ with $u \geq 0$ in $Q_{4\sqrt{n}}$. Applying Proposition 4.1 to u_δ then sending $\delta \longrightarrow 0$, we get

$$\sup_{Q_{1/4}} u \leq C(\inf_{Q_{1/4}} u + \|f\|_{L^n(Q_{4\sqrt{n}})}).$$

Then estimate (4.5) follows from a standard covering argument.

Lemma 4.3. *Suppose u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist constants $\epsilon_0 > 0$, $\mu \in (0, 1)$, and $M > 1$, depending only on n, λ, and Λ, such that if*

$$u \geq 0 \ \text{ in } \ B_{2\sqrt{n}}, \inf_{Q_3} u \leq 1 \ \text{ and } \ \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0, \tag{4.6}$$

then

$$|\{u \leq M\} \cap Q_1| > \mu.$$

Proof. The idea here to localize where the contact set occurs by choosing suitable functions. Namely, we construct a function g that is "very concave" outside Q_1 so that if we "correct" u by g, the contact set is in Q_1. First note that $B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{2\sqrt{n}}$. Define g in $B_{2\sqrt{n}}$ by

$$g(x) = -M(1 - |x|^2/4n)^\beta$$

for some $\beta > 0$ to be specified later and some $M > 0$. We choose M with respect to β so that

$$g \equiv 0 \ \text{ on } \ \partial B_{2\sqrt{n}}, \ \text{ and } \ g \leq -2 \ \text{ in } \ Q_3. \tag{4.7}$$

Set $w = u + g$ in $B_{2\sqrt{n}}$. We shall prove that w, in particular g, belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}} \backslash Q_1$ provided we choose β large enough. Suppose φ is a quadratic polynomial such that $w - \varphi$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$. Then $u - (\varphi - g)$ has a local minimum at x_0 as well. By definitions of $\mathcal{S}^+(\lambda, \Lambda, f)$ and the Pucci extremal operator $\mathcal{M}^-$,

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0) - D^2 g(x_0)) \leq f(x_0),$$

or

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0)) + \mathcal{M}^-(\lambda, \Lambda, -D^2 g(x_0)) \le f(x_0).$$

Therefore, to show g belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}} \backslash Q_1$, it remains to show $\mathcal{M}^-(\lambda, \Lambda, -D^2 g(x_0))$ is non-negative. Well, the Hessian matrix of g is given by

$$D_{ij} g(x) = (M\beta/2n)(1 - |x|^2/4n)^{\beta-1}\delta_{ij} - [M\beta(\beta-1)/(2n)^2](1 - |x|^2/4n)^{\beta-2} x_i x_j.$$

Choose $x = (|x|, 0, 0, \ldots, 0)$, then the eigenvalues of $-D^2 g(x)$ are given by

$$\begin{aligned} e^+ &= (M\beta/2n)(1 - |x|^2/4n)^{\beta-2}((2\beta - 1)|x|^2/4n - 1) \text{ with multiplicity } 1, \\ e^- &= -(M\beta/2n)(1 - |x|^2/4n)^{\beta-2} \text{ with multiplicity } n-1. \end{aligned}$$

Now choose $\beta > 0$ large enough so that $e^+ > 0$ and $e^- < 0$ for $|x| \ge 1/4$. Thus, for $|x| \ge 1/4$, we have

$$\begin{aligned} \mathcal{M}^-(\lambda, \Lambda, -D^2 g(x)) &= \lambda e^+(x) + (n-1)\Lambda e^-(x) \\ &= \frac{M\beta}{2n}(1 - |x|^2/4n)^{\beta-2}\Big[\lambda\Big(\frac{2\beta - 1}{4n}|x|^2 - 1\Big) - (n-1)\Lambda(1 - |x|^2/4n)\Big] \\ &\ge 0. \end{aligned}$$

In fact, we have actually proved that

$$w \in \mathcal{S}^+(\lambda, \Lambda, f + \eta) \ \text{ in } \ B_{2\sqrt{n}}$$

for some $\eta \in C_0^\infty(Q_1)$ and $supp(\eta) \subseteq [0, C(n\lambda, \Lambda)]$. We may apply the Alexandroff maximum principle (Theorem 4.1) to w in $B_{2\sqrt{n}}$. Also note that $\inf_{Q_3} w \le -1$ and $w \ge 0$ on $\partial B_{2\sqrt{n}}$ due to (4.6) and (4.7). Thus,

$$\begin{aligned} 1 &\le C\Big(\int_{B_{2\sqrt{n}} \cap \{w = \Gamma_w\}} (|f| + \eta)^n \, dx\Big)^{1/n} \\ &\le C\|f\|_{L^n(B_{2\sqrt{n}})} + C|\{w = \Gamma_w\} \cap Q_1|^{1/n}. \end{aligned}$$

Choosing ϵ_0 small enough, we get

$$(1/2) \le C|\{w = \Gamma_w\} \cap Q_1|^{1/n} \le C|\{u \le M\} \cap Q_1|^{1/n}$$

since $w(x) = \Gamma_w(x)$ implies $w(x) \le 0$ and thus $u(x) \le -g(x) \le M$. This completes the proof.

□

Next we derive the power decay property of the distribution function of u.

Lemma 4.4. *Let u belong to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist positive constants ϵ_0, ϵ and C, depending only on n, λ, and Λ, such that if*

$$u \geq 0 \text{ in } B_{2\sqrt{n}}, \inf_{Q_3} u \leq 1 \text{ and } \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0, \tag{4.8}$$

then

$$|\{u \geq t\} \cap Q_1| \leq Ct^{-\epsilon} \text{ for } t > 0.$$

Proof. Under the assumptions (4.8), we claim

$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k \text{ for } k = 1, 2, \dots, \tag{4.9}$$

where M and μ are the same parameters from Lemma 4.3. We proceed by induction. Indeed, for $k = 1$, (4.9) is just Lemma 4.3. So assume (4.9) holds for $k - 1$. Set $A = \{u > M^k\} \cap Q_1$ and $B = \{u > M^{k-1}\} \cap Q_1$. We claim that

$$|A| \leq (1-\mu)|B| \tag{4.10}$$

We do so by using Lemma 4.2. Clearly, $A \subset B \subset Q_1$ and $|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu$ by Lemma 4.3. We claim that if $Q = Q_r(x_0)$ is a cube in Q_1 such that

$$|A \cap B| > (1-\mu)|Q|, \tag{4.11}$$

then $\tilde{Q} \cap Q_1 \subset B$ for $\tilde{Q} = Q_{3r}(x_0)$. We prove this by contradiction. Consider the transformation $x = x_0 + ry$ for $y \in Q_1$ and $x \in Q = Q_r(x_0)$, and the function

$$\tilde{u}(y) = M^{-(k-1)}u(x).$$

Then $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\inf_{Q_3} \tilde{u} \leq 1$. It is easy to check that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$. In fact,

$$\tilde{f}(y) = \frac{r^2}{M^{k-1}} f(x) \text{ for } y \in B_{2\sqrt{n}}.$$

Hence,

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{M^{k-1}} \|f\|_{L^n(B_{2\sqrt{n}})} \leq \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0.$$

Therefore, $\tilde{u}$ satisfies (4.8). Thus, Lemma 4.3 applied to $\tilde{u}$ implies

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = r^{-n}|\{u(x) \leq M^k\} \cap Q|.$$

Hence, $|Q \cap A^c| > \mu|Q|$, but this contradicts with (4.11). Applying Lemma 4.2 yields (4.10). $\square$

Proof of Proposition 4.1. We show there exist two constants $\theta > 1$ and $M_0 \gg 1$, depending only on n, λ, and Λ, such that if $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$ there exists a sequence $\{x_k\} \subset B_{1/2}$ such that

$$u(x_k) \geq \theta^k P \text{ for } k = 0, 1, 2, \dots.$$

This contradicts with the boundedness of u and thus $\sup_{B_{1/4}} u \leq M_0$.

Suppose $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$. We will determine M_0 and θ in the process. Consider a cube $Q_r(x_0)$ centered at x_0 with side length r, which will be specified below. We want to find a point $x_1 \in Q_{4\sqrt{n}r}(x_0)$ such that $u(x_1) \geq \theta P$. To do so, we choose r such that $\{u > P/2\}$ covers less than half of $Q_r(x_0)$. This can be done using the power decay of the distribution function of u (see Lemma 4.4). Namely, since $\inf_{Q_3} u \leq \inf_{Q_{1/4}} u \leq 1$, Lemma 4.4 implies

$$|\{u > P/2\} \cap Q_1| \leq C(P/2)^{-\epsilon}.$$

We choose r such that $r^n/2 \geq C(P/2)^{-\epsilon}$ and $r \leq 1/4$. Hence, we have, for such r, $Q_r(x_0) \subset Q_1$ and

$$\frac{1}{|Q_r(x_0)|}|\{u > P/2\} \cap Q_r(x_0)| \leq 1/2. \tag{4.12}$$

Next we show that for $\theta > 1$, with $\theta - 1$ small, $u \geq \theta P$ at some point in $Q_{4\sqrt{n}r}(x_0)$. We proceed by contradiction. That is, assume $u \leq \theta P$ in $Q_{4\sqrt{n}r}(x_0)$. Consider the transformation

$$x = r_0 + ry \text{ for } Q_{4\sqrt{n}} \text{ and } x \in Q_{4\sqrt{n}r}(x_0)$$

and the function

$$\tilde{u}(y) = \frac{\theta P - u(x)}{(\theta - 1)P}.$$

Clearly, $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\tilde{u}(0) = 1$, and thus $\inf_{Q_3} \tilde{u} \leq 1$. It follows that $\tilde{u}$ belongs to $\mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$. Indeed, we have

$$\tilde{f}(y) = -\frac{r^2}{(\theta - 1)P} f(x) \text{ for } y \in B_{2\sqrt{n}}$$

and so

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{(\theta - 1)P}\|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$$

provided we choose P so that $r \leq (\theta - 1)P$. Applying Lemma 4.3 to $\tilde{u}$ and noting that $u(x) \leq P/2 \iff \tilde{u}(y) \geq (\theta - 1/2)/(\theta - 1) \gg 1$ provided that θ is close to 1, we get

$$\begin{aligned} \frac{1}{|Q_r(x_0)|}|\{u \leq P/2\} \cap Q_r(x_0)| &= |\tilde{u} \geq (\theta - 1/2)/(\theta - 1)\} \cap Q_1| \\ &\leq C((\theta - 1/2)/(\theta - 1))^{-\epsilon} < 1/2. \end{aligned}$$

This contradicts with (4.12). Hence, we deduce the existence of a $\theta = \theta(n, \lambda, \Lambda) > 1$ such that if

$$u(x_0) = P \text{ for some } x_0 \in B_{1/4},$$

then

$$u(x_1) \ge \theta P \text{ for some } x_1 \in Q_{4\sqrt{n}r}(x_0) \subset B_{2nr}(x_0)$$

provided that

$$C(n, \lambda, \Lambda)P^{-\epsilon/n} \le r \le (\theta - 1)P.$$

Specifically, we need to choose P such that $P \ge (C/(\theta-1))^{n/(n+\epsilon)}$ and then take $r = CP^{-\epsilon/n}$.

Iterating the previous result yields a sequence $\{x_k\}$ such that of for any $k = 1, 2, 3, \ldots,$

$$u(x_k) \ge \theta^k P \text{ for some } x_k \in B_{2nr_k}(x_{k-1})$$

where $r_k = C(\theta^{k-1}P)^{-\epsilon/n} = C\theta^{-(k-1)\epsilon/n}P^{-\epsilon/n}$.

To ensure $\{x_k\} \subset B_{1/2}$, we take $\sum 2nr_k < 1/4$. Hence, we choose M_0 so that

$$M_0^{\epsilon/n} \ge 8nC \sum_{k=1}^{\infty} \theta^{-(k-1)\epsilon/n} \text{ and } M_0 \ge \Big(\frac{C}{\theta - 1}\Big)^{n/(n+\epsilon)},$$

and choose $P > M_0$. This completes the proof. □

4.3 Schauder Estimates

In this section, we prove the Schauder estimates for viscosity solutions. Throughout this section, we always assume that $a^{ij}(x) \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for any $x \in B_1$ and any $\xi \in \mathbb{R}^n$.

We shall need the following approximation result. Namely, it states that if the coefficient matrix $(a^{ij}(x))$ is a "close" perturbation of the constant matrix $(a^{ij}(0))$ and thus is "close" to the identity matrix by the uniform ellipticity assumption, then the viscosity solution u is "close" to a solution of a Poisson equation at least locally.

Lemma 4.5. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \text{ in } B_1$$

with $|u| \le 1$ in B_1. Assume for some $\epsilon \in (0, 1/16)$,

$$\|a^{ij} - a^{ij}(0)\|_{L^n(B_{3/4})} \le \epsilon.$$

Then there exists a function $h \in C(\bar{B}_{3/4})$ with $a^{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$ for which

$$\|u - h\|_{L^\infty(B_{1/2})} \le C(\epsilon^\gamma + \|f\|_{L^n(B_1)})$$

where $C > 0$ is a constant and $\gamma \in (0, 1)$ both depending only on $n, \lambda,$ and Λ.

Proof. We can certainly solve for such a harmonic function $h \in C(\bar{B}_{3/4}) \cap C^\infty(B_{3/4})$ where $a^{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $h = u$ on $\partial B_{3/4}$. The maximum principle ensures $|h| \leq 1$ in $B_{3/4}$ and note that u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1. Corollary 4.1 implies $u \in C^\alpha(\bar{B}_{3/4})$ for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$. Thus, from the global Schauder regularity theory for harmonic functions, the basic estimate

$$\|u\|_{C^\alpha(\bar{B}_{3/4})} \leq C(n, \lambda, \Lambda)(1 + \|f\|_{L^n(B_1)})$$

implies

$$\|h\|_{C^{\alpha/2}(\bar{B}_{3/4})} \leq C\|u\|_{C^\alpha(\bar{B}_{3/4})} \leq C(n, \lambda, \Lambda)(1 + \|f\|_{L^n(B_1)}).$$

Since $u - h = 0$ on $\partial B_{3/4}$, we get for $\delta \in (0, 1/4)$,

$$\|u - h\|_{L^\infty(\partial B_{3/4-\delta})} \leq C\delta^{\alpha/2}(1 + \|f\|_{L^n(B_1)}). \tag{4.13}$$

We claim that

$$\|D^2 h\|_{L^\infty(\partial B_{3/4-\delta})} \leq C\delta^{\alpha/2-2}(1 + \|f\|_{L^n(B_1)}). \tag{4.14}$$

In fact, for any $x_0 \in B_{3/4-\delta}$, applying interior C^2 estimates on $h - h(x_1)$ in $B_\delta(x_0) \subset B_{3/4}$ for some $x_1 \in \partial B_\delta(x_0)$ yields

$$|D^2 h(x_0)| \leq C\delta^{-2} \sup_{B_\delta(x_0)} |h - h(x_1)| \leq C\delta^{-2}\delta^{\alpha/2}(1 + \|f\|_{L^n(B_1)}).$$

Note that $u - h$ is a viscosity solution of

$$a^{ij}(x)D_{ij}(u - h) = f(x) - (a^{ij}(x) - a^{ij}(0))D_{ij}h := F \text{ in } B_{3/4}.$$

So by the Alexandroff maximum principle and (4.13)-(4.14),

$$\begin{aligned}
\|u - h\|_{L^\infty(B_{3/4-\delta})} &\leq \|u - h\|_{L^\infty(B_{3/4-\delta})} + C\|F\|_{L^n(B_{3/4-\delta})} \\
&\leq \|u - h\|_{L^\infty(B_{3/4-\delta})} + C\|D^2\|_{L^\infty(B_{3/4-\delta})}\|a^{ij} - a^{ij}(0)\|_{L^n(B_{3/4})} + C\|f\|_{L^n(B_1)} \\
&\leq C(\delta^{\alpha/2} + \delta^{\alpha/2-2}\epsilon)(1 + \|f\|_{L^n(B_1)}) + C\|f\|_{L^n(B_1)}.
\end{aligned}$$

The proof is complete once we take $\delta = \sqrt{\epsilon}$ and then $\gamma = \alpha/4$.

□

Definition 4.4. *A function g is Hölder continuous at 0 with exponent α in the L^n sense if*

$$[g]_{C^\alpha_{L^n}}(0) = \sup_{0 \leq r \leq 1} \frac{1}{r^\alpha}\Big(\frac{1}{|B_r|}\int_{B_r} |g(x) - g(0)|^n \, dx\Big)^{1/n} < \infty.$$

Theorem 4.3 (Schauder estimates)**.** *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \text{ in } B_1.$$

Assume (a^{ij}) is Hölder continous at 0 with exponent α in the L^n sense for some $\alpha \in (0,1)$. If f is Hölder continuous at 0 with exponent α in the L^n sense, then u is $C^{2,\alpha}$ at 0. Moreover, there exists a polynomial P of degree 2 such that

$$|u - P|_{L^\infty(B_r(0))} \le C_* r^{2+\alpha} \text{ for any } r \in (0,1),$$
$$|P(0)| + |DP(0)| + |D^2P(0)| \le C_*,$$
$$C_* \le C(\|u\|_{L^\infty(B_1)} + |f(0)| + [f]_{C^\alpha_{L^n}}(0)),$$

where $C > 0$ is a constant depending only on $n, \lambda, \Lambda, \alpha$ and $[a^{ij}]_{C^\alpha_{L^n}}(0)$.

Proof. We organize the proof into two steps.
Step 1: Preparations We assume $f(0) = 0$ otherwise we may consider $v = u - b^{ij}x_ix_jf(0)/2$ for some constant matrix (b^{ij}) such that $a^{ij}(0)b^{ij} = 1$. By scaling, we also assume that $[a^{ij}]_{C^\alpha_{L^n}}(0)$ is small. Next, by considering for $\delta > 0$,

$$\frac{u}{\|u\|_{L^\infty(B_1)} + \delta^{-1}[f]_{C^\alpha_{L^n}}(0)},$$

we may also assume $\|u\|_{L^\infty(B_1)} \le 1$ and $[f]_{C^\alpha_{L^n}}(0) \le \delta$.
Step 2: Suppose $u \in C(B_1)$ is a viscosity solution of

$$a^{ij}(x)D_{ij}u = f \text{ in } B_1$$

with

$$\|u\|_{L^\infty(B_1)} \le 1,\ [a^{ij}]_{C^\alpha_{L^n}}(0) \le \delta$$

and

$$\Big(\frac{1}{|B_r|}\int_{B_r} |f|^n\, dx\Big)^{1/n} \le \delta r^\alpha \text{ for any } r \in (0,1).$$

We claim there exists a constant $\delta > 0$, depending only on n, λ, Λ, and α and a polynomial P of degree 2 with

$$\|u - P\|_{L^\infty(B_r)} \le Cr^{2+\alpha} \text{ for any } r \in (0,1), \tag{4.15}$$

and

$$|P(0)| + |DP(0)| + |D^2P(0)| \le C(n, \lambda, \Lambda, \alpha). \tag{4.16}$$

First, we show there exist $\mu \in (0,1)$, depending only on n, λ, Λ, and α, and a sequence of polynomials of degree 2,

$$P_k(x) = a_k + b_k \cdot x + (1/2)x^TC_kx,$$

such that for any $k = 0, 1, 2, \dots,$

$$a^{ij}(0)D_{ij}P_k = 0,\ \|u - P_k\|_{L^\infty(B_{\mu^k})} \le \mu^{k(2+\alpha)}, \tag{4.17}$$

and

$$|a_k - a_{k-1}| + \mu^{k-1}|b_k - b_{k-1}| + \mu^{2(k-1)}|C_k - C_{k-1}| \le C\mu^{(k-1)(2+\alpha)}. \tag{4.18}$$

Note that $P_0, P_{-1} \equiv 0$ and C is a constant depending only on n, λ, Λ, and α.

Obviously, the theorem follows from (4.17)-(4.18) since a_k, b_k and C_k converge to some a, b and C, and the limiting polynomial,

$$P(x) = a + b \cdot x + (1/2)x^T Cx,$$

satisfies

$$|P_k(x) - P(x)| \le C(|x|^2\mu^{\alpha k} + |x|\mu^{(\alpha+1)k} + \mu^{(\alpha+2)k}) \le C\mu^{(2+\alpha)k}$$

for any $|x| \le \mu^k$. Hence, for $|x| \le \mu^k$,

$$|u(x) - P(x)| \le |u(x) - P_k(x)| + |P_k(x) - P(x)| \le C\mu^{(2+\alpha)k},$$

which implies

$$|u(x) - P(x)| \le C|x|^{2+\alpha} \text{ for any } x \in B_1.$$

Therefore, it only remains to prove (4.17) and (4.18), and we do so by induction. The initial step $k = 0$ is clearly true. Assume both estimates hold for $k = 0, 1, \dots, \ell$. We prove the next step $k = \ell + 1$ holds. Consider the function

$$\tilde{u}(y) = \frac{1}{\mu^{\ell(2+\alpha)}}(u - P_\ell)(\mu^\ell y) \text{ for } y \in B_1.$$

Then $\tilde{u}$ belongs to $C(B_1)$ and is a viscosity solution of

$$\tilde{a}^{ij}(x)D_{ij}\tilde{u} = \tilde{f} \text{ in } B_1$$

where

$$\tilde{a}^{ij}(y) = \mu^{-\ell\alpha}a^{ij}(\mu^\ell y),$$

and

$$\tilde{f}(y) = \mu^{-\ell\alpha}(f(\mu^\ell y) - a^{ij}(\mu^\ell y)D_{ij}P_k).$$

We want to apply Lemma 4.16. So we check that

$$\|\tilde{a}^{ij} - \tilde{a}^{ij}(0)\|_{L^n(B_1)} \le \mu^{-\ell\alpha}\|a^{ij} - a^{ij}(0)\|_{L^n(B_{\mu^\ell})} \le [a^{ij}]^\alpha_{L^n}(0) \le \delta,$$

and

$$\|\tilde{f}\|_{L^n(B_1)} \le \mu^{-\ell\alpha}\|f\|_{L^n(B_{\mu^\ell})} + \mu^{-\ell\alpha}\sup|D^2P_\ell|\|a^{ij} - a^{ij}(0)\|_{L^n(B_{\mu^\ell})} \le \delta + C\delta$$

where we used

$$|D^2P_\ell| \le \sum_{k=1}^{\ell}|D^2P_k - D^2P_{k-1}| \le \sum_{k=1}^{\ell}\mu^{(k-1)\alpha} \le C.$$

Taking $\epsilon = C(n, \lambda, \Lambda)\delta$ in Lemma 4.16, we can find $h \in C(\bar{B}_{3/4})$ with $\tilde{a}^{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \leq 1$ in $B_{3/4}$ such that

$$\|\tilde{u} - h\|_{L^\infty(B_{1/2})} \leq C(\epsilon^\gamma + \epsilon) \leq 2C\epsilon^\gamma.$$

Write $\tilde{P}(y) = h(0) + Dh(0) + y^T D^2 h(0) y/2$. Then the interior estimates for h yield

$$\|\tilde{u} - \tilde{P}\|_{L^\infty(B_\mu)} \leq \|\tilde{u} - h\|_{L^\infty(B_\mu)} + \|h - \tilde{P}\|_{L^\infty(B_\mu)} \leq 2C\epsilon^\gamma + C\mu^3 \leq \mu^{2+\alpha}$$

by choosing μ small and then ϵ small accordingly. Rescaling back, we get

$$|u(x) - P_\ell(x) - \mu^{\ell(2+\alpha)}\tilde{P}(\mu^{-\ell}x)| \leq \mu^{(\ell+1)(2+\alpha)} \text{ for any } x \in B_{\mu^{\ell+1}}.$$

This implies (4.17) for $k = \ell + 1$ if we take

$$P_{k+1}(x) = P_k(x) + \mu^{\ell(2+\alpha)}\tilde{P}(\mu^{-\ell}x).$$

Estimate (4.18) follows easily. □

We also have the following Cordes-Nirenberg type estimate, but we omit its proof.

Theorem 4.4 (Cordes-Nirenberg). *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \text{ in } B_1.$$

Then for any $\alpha \in (0,1)$, there exists an $\theta > 0$ depending only on n, λ, Λ, and α such that if

$$\Big(\frac{1}{|B_r|}\int_{B_r} |a^{ij}(x) - a^{ij}(0)|^n\, dx\Big)^{1/n} \leq \theta \text{ for any } f \in (0,1),$$

then u is $C^{1,\alpha}$ at 0. Namely, there exists an affine function L such that

$$\begin{aligned}
&|u - L|_{L^\infty(B_r(0))} \leq C_* r^{1+\alpha} \text{ for any } r \in (0,1),\\
&|L(0)| + |DL(0)| \leq C_*,\\
&C_* \leq C\Big\{\|u\|_{L^\infty(B_1)} + \sup_{0<r<1}\Big(\frac{1}{|B_r|}\int_{B_r} |f(x)|^n\, dx\Big)^{1/n}\Big\},
\end{aligned}$$

where $C > 0$ is a constant depending only on n, λ, Λ, and α.

4.4 $W^{2,p}$ Estimates

In this section, we assume throughout that $f \in C(B_1)$, $(a^{ij}) \in C(B_1)$ and there exist $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for any $x \in U$ and any $\xi \in \mathbb{R}^n$. Our main result here is the following

Theorem 4.5. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \ \text{ in } \ B_1.$$

For any $p \in (n, \infty)$, there exists an $\epsilon > 0$ depending only on n, λ, Λ, and p such that if

$$\Big(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |a^{ij}(x) - a^{ij}(x_0)|^n\, dx\Big)^{1/n} \le \epsilon \ \text{ for any } B_r(x_0) \subset B_1,$$

then $u \in W^{2,p}_{loc}(B_1)$. Moreover,

$$\|u\|_{W^{2,p}(B_{1/2})} \le C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}),$$

where $C > 0$ is a constant depending only on n, λ, Λ, and p.

As before, it suffices to prove the following.

Theorem 4.6. *Suppose $u \in C(B_{8\sqrt{n}})$ is a viscosity solution of*

$$a^{ij}(x)D_{ij}u = f \ \text{ in } \ B_{8\sqrt{n}}.$$

For any $p \in (n, \infty)$, there exist $\epsilon > 0$ and $C > 0$ depending only on n, λ, Λ, and p such that if

$$\|u\|_{L^\infty(B_{8\sqrt{n}})} \le 1 \ \text{ and } \ \|f\|_{L^p(B_{8\sqrt{n}})} \le \epsilon$$

and if

$$\Big(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |a^{ij}(x) - a^{ij}(x_0)|^n\, dx\Big)^{1/n} \le \epsilon \ \text{ for any } B_r(x_0) \subset B_{8\sqrt{n}},$$

then $u \in W^{2,p}(B_1)$ and $\|u\|_{W^{2,p}(B_1)} \le C$.

CHAPTER 5

The Method of Moving Planes and Its Variants

In this chapter, we introduce a powerful tool used to study the properties of solutions for semilinear elliptic equations. The method is called the method of moving planes and it originated from Alexandroff in his study of embedded constant mean curvature surfaces. It was further developed in the works of Serrin [22] and Gidas, Ni and Nirenberg [10] and later adapted to many other problems involving differential and integral equations (see [5] and the references therein). We will focus on applying this method to obtain symmetry and monotonicity results for positive solutions of the Lane-Emden equation and we shall essentially adopt the framework of Chen and Li [4].

Consider the following semilinear elliptic problem

$$-\Delta u = u^p, \ \ x \in \mathbb{R}^n, \ \ \ n \geq 3. \tag{5.1}$$

Our goal is to prove the following main result.

Theorem 5.1. *For $p = (n+2)/(n-2)$, every positive C^2 solution of equation* (5.1) *must be radially symmetric and monotone decreasing about some point, and thus assumes the form*

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^0|^2)^{\frac{n-2}{2}}} \ \text{ for some } \ \lambda > 0 \ \text{ and } \ x^0 \in \mathbb{R}^n.$$

For $1 < p < (n+2)/(n-2)$, the only non-negative C^2 solution of equation (5.1) *is the trivial one, $u \equiv 0$.*

5.1 Preliminaries

We first start by introducing some necessary tools for the method of moving planes. Namely, we introduce the Kelvin transform and various comparison theorems, i.e., maximum principles, for elliptic problems on unbounded domains. First, the Kelvin transform of the function u, which we denote by $\bar{u}$, is given by

$$\bar{u}(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big).$$

Then, if u is a solution of equation (5.1), then $\bar{u}$ is a solution of

$$-\Delta \bar{u} = |x|^{p(n-2)-(n+2)} \bar{u}^p, \quad x \in \mathbb{R}^n \backslash \{0\}. \tag{5.2}$$

Now we introduce the maximum principles based on comparisons, which are the essential ingredients in the method of moving planes.

Theorem 5.2. *Assume that U is a bounded domain. Let ϕ be a positive function on $\bar{U}$ satisfying*

$$-\Delta \phi + \lambda(x)\phi \geq 0. \tag{5.3}$$

Assume that u is a solution of

$$\begin{cases} -\Delta u + c(x)u \geq 0 & \text{in } U, \\ u \geq 0 & \text{on } \partial U. \end{cases} \tag{5.4}$$

If

$$c(x) > \lambda(x) \;\; \text{for all} \;\; x \in U, \tag{5.5}$$

then

$$u \geq 0 \;\; \text{in} \;\; U.$$

If U is unbounded, then the result remains true provided that the following additional condition is assumed:

$$\liminf_{|x| \longrightarrow \infty} \frac{u(x)}{\phi(x)} \geq 0. \tag{5.6}$$

Proof. We proceed by contradiction. Let $v(x) = u(x)/\phi(x)$ and assume that $u < 0$ at some point in U. Thus, $v < 0$ at that same point, since ϕ is positive in U. Let $x^0 \in U$ be the minimum of v and by a simple calculation, we obtain that

$$-\Delta v = 2Dv \cdot \frac{D\phi}{\phi} + \frac{1}{\phi}(-\Delta u + \frac{\Delta \phi}{\phi} u). \tag{5.7}$$

However, since x^0 is a minimum of v, we have that

$$-\Delta v(x^0) \leq 0 \tag{5.8}$$

and

$$Dv(x^0) = 0. \tag{5.9}$$

But from (5.3)–(5.5) and since $u(x^0) < 0$, we have that

$$-\Delta u(x^0) + \frac{\Delta \phi}{\phi}(x^0)u(x^0) \geq -\Delta u(x^0) + \lambda(x^0)u(x^0) > -\Delta u(x^0) + c(x^0)u(x^0) \geq 0.$$

By inserting this into (5.7) and using (5.9), we get that $-\Delta v(x^0) > 0$, but this contradicts with (5.8). This completes the proof. In the case that U is unbounded, the same arguments apply since the additional assumption (5.6) guarantees that the minimum of v does not leak away to infinity. □

Remark 5.1. *As illustrated in the proof, conditions* (5.3) *and* (5.5) *are required only at the points where v attains its minimum or at points where u is negative.*

In our application of the above theorem, we will consider two cases:

(a) U is a "narrow" region,

(b) the coefficient $c(x)$ has sufficient decay at infinity.

First, we examine when U is a narrow region; namely, let us consider the narrow strip with width $\ell > 0$, i.e.,

$$U = \{x \in \mathbb{R}^n \,|\, 0 < x_1 < \ell\}.$$

We can take $\varphi(x) = \sin((x_1 + \epsilon)/\ell)$ so that $-\Delta\varphi = (1/\ell)^2\varphi$. Thus, $\lambda(x) = -(1/\ell)^2$, which can be "very negative" if ℓ is suitably small.

Corollary 5.1 (Narrow region)**.** *If u satisfies* (5.4) *with bounded function $c(x)$, the width ℓ of the region U is sufficiently small, $c(x)$ satisfies* (5.5)*, i.e., $c(x) > \lambda(x) = -1/\ell^2$, then*

$$u \geq 0 \ \text{ in } U.$$

In the case of (b) with $n \geq 3$, we can choose a positive number $q < n - 2$ and take $\phi(x) = |x|^{-q}$, then a simple calculation yields

$$-\Delta\phi = \frac{q(n-2-q)}{|x|^2}\phi := -\lambda(x)\phi.$$

Therefore, if $c(x)$ has sufficient decay, the previous theorem implies the following.

Corollary 5.2 (Decay at infinity)**.** *Assume there exists $R > 0$ such that*

$$c(x) > -\frac{q(n-2-q)}{|x|^2}, \ \text{ for all } |x| > R. \tag{5.10}$$

Suppose that

$$\lim_{|x|\to\infty} u(x)|x|^q = 0.$$

Let U be a region contained in $B_R^C(0)$. If u satisfies (5.4) *on $\bar{U}$, then*

$$u(x) \geq 0 \ \text{ for all } x \in U.$$

Remark 5.2. *As pointed out in the last remark, one can see that condition* (5.10) *is only required at points where u is negative.*

5.2 The Proof of Theorem 5.1

We are now ready to prove Theorem 5.1.

Proof. Set $p = \frac{n+2}{n-2}$ and we shall first impose a fast decay assumption on the solution, i.e.,

$$u(x) = O(|x|^{-(n-2)}). \tag{5.11}$$

Define

$$\Sigma_\lambda := \Big\{ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \,|\, x_1 < \lambda \Big\} \text{ and } T_\lambda := \partial \Sigma_\lambda$$

and let x^λ be the reflection point of x about the plane T_λ, i.e.,

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n).$$

Define

$$u_\lambda(x) := u(x^\lambda) \text{ and } w_\lambda(x) := u_\lambda(x) - u(x).$$

Step 1: Prepare to move the plane near $-\infty$.

Namely, we will show that we can find $N > 0$ suitably large so that if $\lambda \leq -N$,

$$w_\lambda(x) \geq 0 \text{ for all } x \in \Sigma_\lambda. \tag{5.12}$$

Indeed, the mean value theorem implies

$$-\Delta w_\lambda(x) = u_\lambda^p(x) - u^p(x) = p\psi_\lambda^{p-1} w_\lambda(x), \tag{5.13}$$

where $\psi_\lambda(x)$ is some number between $u_\lambda(x)$ and $u(x)$. In view of Theorem 5.2 and Corollary 5.2, we take $c(x) = -p\psi_\lambda^{p-1}(x)$ and see that (5.12) holds provided we show $c(x)$ has sufficient decay at infinity at the points $\tilde{x}$ where $w_\lambda(\tilde{x}) < 0$. Well, at these points, we have

$$u_\lambda(\tilde{x}) < u(\tilde{x})$$

and so

$$0 \leq u_\lambda(\tilde{x}) \leq \psi_\lambda(\tilde{x}) \leq u(\tilde{x}).$$

Indeed, by assumption (5.11) and since $p = \frac{n+2}{n-2}$,

$$\psi_\lambda^{p-1}(\tilde{x}) = O\Big((|\tilde{x}|^{-(n-2)})^{\frac{4}{n-2}} \Big) = O(|\tilde{x}|^{-4})$$

and the decay of the coefficient is greater than 2 as required in Corollary 5.2, which implies the desired result. Namely, we can find $N > 0$ sufficiently large so that for $\lambda \leq -N$ (or $|\tilde{x}|$ sufficiently large), we must have (5.12).

Step 2: Moving the Plane.

We can increase the value of λ, and thus move the plane T_λ to right, provided inequality (5.12) holds. Define

$$\lambda_0 := \sup\{\lambda \,|\, w_\lambda(x) \geq 0, \text{ for all } x \in \Sigma_\lambda\}.$$

Clearly, $\lambda_0 < \infty$ due to the asymptotic behavior of u for x_1 near ∞. We claim that

$$w_{\lambda_0} \equiv 0 \text{ in } \Sigma_{\lambda_0}. \tag{5.14}$$

Otherwise, the strong maximum principle on unbounded domains implies that

$$w_{\lambda_0}(x) > 0 \text{ for all } x \in interior(\Sigma_{\lambda_0}). \tag{5.15}$$

Then we show that we can then move the plane T_{λ_0} further to the right a small distance, thereby contradicting the definition of λ_0 and conclude that (5.14) holds. Namely, we claim there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, we have that

$$w_{\lambda_0+\delta}(x) \geq 0 \text{ for all } x \in \Sigma_{\lambda_0+\delta}. \tag{5.16}$$

At first glance, one may assume that this would follow from Corollary 5.1, however, we cannot apply this directly since we are not able to guarantee that w_{λ_0} is bounded away from 0 on the left boundary of the narrow region. To circumvent this, we apply Corollary 5.2 instead but to a carefully chosen auxiliary function. Namely, we set

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi(x)},$$

where

$$\phi(x) = |x|^{-q} \text{ with } q \in (0, n-2).$$

Then, a direct calculation will show that

$$-\Delta \bar{w}_\lambda = 2D\bar{w}_\lambda \cdot \frac{D\phi}{\phi} + \Big(-\Delta w_\lambda + \frac{\Delta \phi}{\phi} w_\lambda \Big) \frac{1}{\phi}. \tag{5.17}$$

Claim: There exists $R_0 > 0$, independent of λ, such that if x^0 is a minimum point of $\bar{w}_\lambda$ and $\bar{w}_\lambda(x^0) < 0$, then $|x^0| < R_0$.

To show this claim holds, we proceed by contradiction. Assume that x^0 is a negative minimum of $\bar{w}_\lambda$ but that $|x^0|$ can be chosen to be suitably large. Thus,

$$-\Delta \bar{w}_\lambda(x^0) \leq 0, \tag{5.18}$$

and

$$D\bar{w}_\lambda(x^0) = 0. \tag{5.19}$$

By the asymptotic behavior of u at infinity and since $|x^0|$ is sufficiently large,

$$c(x^0) := -p\psi_\lambda(x^0)^{p-1} > -\frac{q(n-2-q)}{|x^0|^2} \equiv \frac{\Delta\phi(x^0)}{\phi(x^0)}.$$

It follows from (5.13) and $w_\lambda(x_0) < 0$ that

$$0 = -\Delta w_\lambda(x^0) + c(x^0) w_\lambda(x^0) < -\Delta w_\lambda(x^0) + \frac{\Delta\phi(x^0)}{\phi(x^0)} w_\lambda(x^0).$$

Hence,

$$\Big(-\Delta w_\lambda + \frac{\Delta\phi}{\phi} w_\lambda \Big)(x^0) > 0.$$

Combining this with (5.17) and (5.19) leads to $-\Delta \bar{w}_\lambda(x^0) > 0$, which contradicts with (5.18). This completes the proof of the claim.

Hence, if (5.16) is violated for any $\delta > 0$, then we can find a sequence of positive numbers $\{\delta_i\} \longrightarrow 0$ where for each i, we denote the corresponding negative minimum of $\bar{w}_{\lambda_0+\delta_i}$ by x^i. Then, by the last claim, we have $|x^i| \leq R_0$ for $i = 1, 2, 3, \ldots$. Then, by compactness, we can extract a subsequence, which we still denote by $\{x^i\}$, that converges to some point $x^0 \in \mathbb{R}^n$. Hence,

$$D\bar{w}_{\lambda_0}(x^0) = \lim_{i\to\infty} D\bar{w}_{\lambda_0+\delta_i}(x^i) = 0,$$

$$\bar{w}_{\lambda_0}(x^0) = \lim_{i\to\infty} \bar{w}_{\lambda_0+\delta_i}(x^i) \leq 0.$$

From this, we deduce that $\bar{w}_{\lambda_0}(x^0) = 0$, since we also know that $\bar{w}_{\lambda_0} \geq 0$. Moreover,

$$D w_{\lambda_0}(x^0) = D\bar{w}_{\lambda_0}(x^0)\phi(x^0) + \bar{w}_{\lambda_0}(x^0) D\phi(x^0) = 0. \tag{5.20}$$

In view of (5.15) and the fact that $w_{\lambda_0}(x^0) = 0$, we must have that x^0 lies on the boundary of Σ_{λ_0}. Then Hopf's lemma indicates that

$$\frac{\partial w_{\lambda_0}}{\partial \nu}(x^0) < 0,$$

which contradicts with (5.20) and we conclude that $w_{\lambda_0} \equiv 0$ or that $u(x) = u_{\lambda_0}(x)$ for all $x \in \Sigma_{\lambda_0}$.

So far, we have shown that u is symmetric and monotone decreasing about the plane T_{λ_0}. Since the coordinate axis x_1 can be chosen arbitrarily, we conclude that u must be radially symmetric and monotone decreasing about some point. Moreover, basic uniqueness theory for ordinary differential equations imply that u must have the form as described in the theorem.

Step 3: Removing the fast decay assumption.

Apply the Kelvin transform on the solution $u(x)$ to get $v(x)$:

$$v(x) = \frac{1}{|x|^{n-2}} u\Big(\frac{x}{|x|^2}\Big).$$

Then v has the fast decay at infinity and satisfies the following semilinear equation in punctured space,

$$-\Delta v = v^p \ \text{ in } \ \mathbb{R}^n \backslash \{0\}.$$

We can apply the same arguments of Steps 1 and 2, after minor modifications (we must carry out the procedure on $\Sigma_\lambda \backslash \{x^\lambda\}$ to avoid the possible singularity introduced by the Kelvin transform) to conclude that v is radially symmetric and monotone decreasing about some point x^0 in $\mathbb{R}^n$. If x^0 is not the origin, then the origin is a regular point and u has the fast decay property at infinity and we are done. Otherwise, if v is symmetric and monotone about the origin, then u is also symmetric and monotone about the origin since it is easy to check that $u(x) = |x|^{-(n-2)} v(x/|x|^2)$.

Step 4: Liouville property in the subcritical case.

It remains to prove that $u \equiv 0$ in the subcritical case $p \in (1, \frac{n+2}{n-2})$. Again, by the Kelvin transform, we have that v, as defined earlier, is now a solution of

$$-\Delta v = |x|^{p(n-2)-(n+2)} v^p \ \text{ in } \ \mathbb{R}^n \backslash \{0\}. \tag{5.21}$$

Since the subcritical condition implies that $p(n-2)-(n+2) < 0$, the coefficient of equation (5.21) decays at infinity. Therefore, we may apply the method of moving planes, i.e., Steps 1–3, to get that v is radially symmetric and monotone decreasing about some point $x^0 \in \mathbb{R}^n$. In fact, it is clear that $x^0 = 0$ due to the singular coefficient of equation (5.21). Thus, it is easy to see that u is also radially symmetric and monotone decreasing about the origin. Then, as a consequence of the well-known Pohozaev type identity for equation (5.21), $u \equiv 0$. Alternatively, we can argue, using the translation and dilation invariance of equation (5.21), that v must actually be constant and therefore trivial. This completes the proof of the theorem. $\square$

Remark 5.3. *We see that the "decay at infinity" principle is important in applying the method of moving planes to the Lane-Emden equation in $\mathbb{R}^n$, but we did not make use of the "narrow region" principle. Indeed, the narrow region principle is more appropriate for certain bounded domains. Namely, it is a key ingredient in applying the method of moving planes for radially symmetric, bounded domains. A consequence of this is the following result whose proof we omit.*

Theorem 5.3. *Assume that f is a Lipschitz continuous function such that*

$$|f(p) - f(q)| \leq C_0 |p - q|$$

for some positive constant C_0. Then every positive solution $u \in C^2(B_1(0)) \cap C(\bar{B}_1(0))$ of

$$\begin{cases} -\Delta u = f(u) & \text{in } B_1(0), \\ \quad u = 0 & \text{on } \partial B_1(0), \end{cases}$$

is radially symmetric and monotone decreasing about the origin.

5.3 The Method of Moving Spheres

In this section, we introduce a variant of the method of moving planes known as the method of moving spheres. This alternative technique uses the inversion of the Kelvin transform on spheres and invokes comparison theorems to obtain symmetry and monotonicity properties of solutions to certain elliptic problems. The advantage of this approach is that we can deduce the classification and Liouville theorems for non-negative solutions in one fell swoop. This is, in some sense, more direct than the method of moving planes, which first establishes the radial symmetry and monotonicity properties then reduces the problem into an ODE one to arrive at the desired results. The moving sphere approach is also advantageous in certain domains such as half-spaces.

First, we state and prove two fundamental calculus lemmas that are important ingredients in the method of moving spheres.

Lemma 5.1. *Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1$ and $\nu > 0$. Suppose that for each $x \in \mathbb{R}^n$, there exists $\lambda = \lambda(x)$ such that*

$$\Big(\frac{\lambda(x)}{|y-x|}\Big)^{\nu} f\Big(x + \lambda(x)^2 \frac{y-x}{|y-x|^2}\Big) = f(y), \quad y \in \mathbb{R}^n \backslash \{x\}. \tag{5.22}$$

Then for some $a \geq 0$, $d > 0$, and $x_0 \in \mathbb{R}^n$,

$$f(x) = \pm \Big(\frac{a}{d + |x-x_0|^2}\Big)^{\nu/2}.$$

Proof. From (5.22), we have that

$$\ell := \lim_{|y| \to \infty} |y|^{\nu} f(y) = \lambda(x)^{\nu} f(x), \quad x \in \mathbb{R}^n.$$

If $\ell = 0$, then $f \equiv 0$ and we are done. However, if $\ell \neq 0$, then f does not change sign. Therefore, without loss of generality, we may take $\ell = 1$ and f positive. For large y, taking Taylor expansions of the left-hand side of (5.22) at 0 and x yield

$$f(y) = \Big(\frac{\lambda(0)}{|y|}\Big)^{\nu} \Big(f(0) + \frac{\partial f}{\partial y_i}(0)\lambda(0)^2 \frac{y_i}{|y|^2} + o(|y|^{-1})\Big) \tag{5.23}$$

and

$$f(y) = \Big(\frac{\lambda(x)}{|y-x|}\Big)^{\nu} \Big(f(x) + \frac{\partial f}{\partial y_i}(x)\lambda(x)^2 \frac{y_i - x_i}{|y-x|^2} + o(|y|^{-1})\Big), \tag{5.24}$$

where $o(|y|^{-1})$ represents some higher-order term such that $o(|y|^{-1})/|y|^{-1} \longrightarrow 0$ as $|y| \longrightarrow \infty$. From our assumption that $\ell = 1$, we combine (5.22), (5.23), and (5.24) together to get

$$f(x)^{-1-2/\nu} \frac{\partial f}{\partial y_i}(x) = f(0)^{-1-2/\nu} \frac{\partial f}{\partial y_i}(0) - \nu x_i.$$

It follows that for some $x_0 \in \mathbb{R}^N$, $d > 0$,

$$f(y)^{-2/\nu} = |y - x_0|^2 + d.$$

Solving for $f(y)$ will finish the proof. $\square$

Lemma 5.2. *Let $f \in C^1(\mathbb{R}^N)$, $n \geq 1$, and $\nu > 0$. Suppose that*

$$\Big(\frac{\lambda}{|y-x|}\Big)^{\nu} f\Big(x + \lambda \frac{y-x}{|y-x|^2}\Big) \leq f(y), \ \ \textit{for all} \ \ \lambda > 0, \, x \in \mathbb{R}^n, \, |y-x| \geq \lambda. \tag{5.25}$$

Then $f \equiv$ constant.

Proof. For $x \in \mathbb{R}^n$, $\lambda > 0$, define

$$g_{x,\lambda}(z) = f(x+z) - \Big(\frac{\lambda}{|z|}\Big)^{\nu} f\Big(x + \lambda^2 \frac{z}{|z|^2}\Big), \ \ |z| \geq \lambda.$$

Indeed, $g_{x,|z|}(z) = 0$ and $g_{x,|z|}(rz) \geq 0$ for $r \geq 1$. Then, it follows that

$$\frac{d}{dr} g_{x,|z|}(rz)\Big|_{r=1} \geq 0.$$

Hence, a direct calculation yields

$$2Df(z+x)\cdot z + \nu f(z+x) \geq 0.$$

Since z and x are chosen arbitrarily, a change of variables shows that

$$2Df(y)\cdot(y-x) + \nu f(y) \geq 0.$$

Multiplying the preceding inequality by $|x|^{-1}$ and sending $|x| \longrightarrow \infty$, we conclude that $Df(y)\cdot\theta \leq 0$ for all $y \in \mathbb{R}^n$ and $\theta \in \mathbb{S}^{n-1}$. Hence, $Df \equiv 0$ in $\mathbb{R}^n$, and this completes the proof. $\square$

We give an alternative proof of Theorem 5.1 using the method of moving spheres. We interrupt momentarily for some notation. For $x \in \mathbb{R}^n$ and $\lambda > 0$, define the Kelvin transformation of u by

$$u_{x,\lambda}(y) = \Big(\frac{\lambda}{|y-x|}\Big)^{n-2} u\Big(x + \lambda^2 \frac{y-x}{|y-x|^2}\Big), \ \ y \in \mathbb{R}^n \backslash \{x\}.$$

The following lemma ensures that we may start the moving sphere procedure.

Lemma 5.3. *For every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that $u_{x,\lambda(x)}(y) \leq u(y)$.*

From this we may define the following value $\lambda_0 \in (0,\infty]$. For each $x \in \mathbb{R}^n$ we set

$$\lambda_0(x) = \sup\{\mu > 0 \,|\, u_{x,\lambda}(y) \leq u(y), \ \ \text{for all} \ \ |y-x| \geq \lambda, \, \lambda \in (0,\mu]\}.$$

Proof of Theorem 5.1. We consider the two cases separately.
Critical case: Let $p = (n+2)/(n-2)$ and suppose that u is a positive solution of (5.1).

Step 1: We claim that if $\lambda_0(x) < \infty$ for some point $x \in \mathbb{R}^n$, then

$$u_{x,\lambda_0(x)} \equiv u \;\text{ in }\; \mathbb{R}^n \backslash \{0\}.$$

Without loss of generality, we may take $x = 0$ and $\lambda_0 = \lambda_0(0)$, $u_\lambda = u_{0,\lambda}$, and

$$\Sigma_\lambda = \{y \in \mathbb{R}^n \mid |y| > \lambda\}.$$

From the definition of λ_0,

$$u \geq u_{\lambda_0} \;\text{ on }\; \Sigma_{\lambda_0}.$$

Recall that the Kelvin transform of u satisfies

$$-\Delta u_\lambda = u_\lambda^{\frac{n+2}{n-2}}, \;\; \lambda > 0.$$

So by setting $w_\lambda = u - u_\lambda$, we get

$$-\Delta w_{\lambda_0} = u^{\frac{n+2}{n-2}} - u_{\lambda_0}^{\frac{n+2}{n-2}} \geq 0 \;\text{ in }\; \Sigma_{\lambda_0}.$$

If $w_{\lambda_0} \equiv 0$ in Σ_{λ_0}, then we are done. Otherwise, Hopf's lemma and the compactness of $\partial B_{\lambda_0}(0)$ imply that

$$\frac{d}{dr} w_{\lambda_0}\Big|_{\partial B_{\lambda_0}(0)} \geq c > 0.$$

By the continuity of Du, there exists $R \geq \lambda_0$ such that

$$\frac{d}{dr} w_\lambda \geq c/2 > 0, \;\text{ for }\; \lambda \in [\lambda_0, R],\, r \in [\lambda, R].$$

Thus, since $w_\lambda \equiv 0$ on $\partial B_\lambda(0)$, we have

$$w_\lambda(y) > 0 \;\text{ for }\; \lambda \in [\lambda_0, R],\, |y| \in (\lambda, R]. \tag{5.26}$$

Setting $m = \min_{\partial B_R(0)} w_{\lambda_0} > 0$ and since $-\Delta w_{\lambda_0} > 0$ in Σ_{λ_0},

$$w_{\lambda_0}(y) \geq m \frac{R^{n-2}}{|y|^{n-2}}, \;\text{ for }\; |y| \geq R.$$

Hence,

$$w_\lambda(y) \geq m \frac{R^{n-2}}{|y|^{n-2}} - (u_\lambda(y) - u_{\lambda_0}(y)), \;\text{ for }\; |y| \geq R. \tag{5.27}$$

By the uniform continuity of u on $\bar{B}_R(0)$, there exists $\epsilon \in (0, R - \lambda_0)$ such that for $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$,

$$\Big|\lambda^{n-2} u\Big(\lambda^2 \frac{y}{|y|^2}\Big) - \lambda_0^{n-2} u\Big(\lambda_0^2 \frac{y}{|y|^2}\Big)\Big| < \frac{mR}{2}, \;\text{ for }\; |y| \geq R.$$

From this and (5.27), we get

$$w_\lambda(y) > 0 \;\text{ for }\; \lambda \in [\lambda_0, \lambda_0 + \epsilon],\; |y| \geq R. \tag{5.28}$$

However, estimates (5.26) and (5.28) contradict the definition of λ_0. This proves the claim.

Step 2: We claim that if $\lambda_0(x_0) = \infty$ for some $x_0 \in \mathbb{R}^n$, then $\lambda_0(x) = \infty$ for all $x \in \mathbb{R}^n$.

Observe that, by definition,

$$u_{x_0,\lambda}(y) \leq u(y), \quad \text{for all } \lambda > 0,\; |y - x_0| \geq \lambda.$$

Thus,

$$\lim_{|y|\to\infty} |y|^{2-n} u(y) = \infty.$$

Assume that $\lambda_0(x) = \infty$ for some $x \in \mathbb{R}^n$. Then by Step 1,

$$\lim_{|y|\to\infty} |y|^{n-2} u(y) = \lim_{|y|\to\infty} |y|^{n-2} u_{x,\lambda_0(x)}(y) = \lambda_0(x)^{n-2} u(x) < \infty,$$

and we arrive at a contradiction.

Step 3: We claim $\lambda_0(x) < \infty$ for all $x \in \mathbb{R}^n$.

To see this, note that if $\lambda_0(x_0) = \infty$ for some point $x_0 \in \mathbb{R}^n$, then Step 2 ensures $\lambda_0(x) = \infty$ for all $x \in \mathbb{R}^n$. Lemma 5.2 then implies that $u \equiv$ *constant*. Since u is assumed to be positive and we have shown it is necessarily constant, we arrive at a contradiction.

Step 4: We are now ready to complete the proof of the theorem in the critical case. From the previous steps, for each $x \in \mathbb{R}^n$ it follows that $\lambda_0(x) < \infty$ and $u_{x,\lambda_0(x)} \equiv u$ in $\mathbb{R}^n \backslash \{x\}$. From Lemma 5.1, there are $a, d > 0$ and some $x_0 \in \mathbb{R}^n$ such that

$$u(x) = \Big(\frac{a}{d + |x - x_0|^2}\Big)^{\frac{n-2}{2}},\; x \in \mathbb{R}^n.$$

This proves the result in the critical case.

Subcritical case: Let $p < (n+2)/(n-2)$ and suppose u is a non-negative solution of (5.1). The proof in this case is similar to the critical case. Namely, due to the Kelvin transform, we can show that $\lambda_0(x_0) = \infty$ for some $x_0 \in \mathbb{R}^n$. As before, this implies that $\lambda_0(x) = \infty$ for each $x \in \mathbb{R}^n$. Then, by Lemma 5.2, $u \equiv$ *constant* and so $u \equiv 0$. This completes the proof. $\square$

CHAPTER 6

Concentration and Non-compactness of Critical Sobolev Embeddings

6.1 Introduction

In this chapter, we explore the breakdown of the compactness of the injection

$$W^{1,p}(U) \hookrightarrow L^q(U)$$

where $1/q = 1/p - 1/n$ (see the appendix for the statements and proofs of the Sobolev inequalities and embeddings). A closely related and important issue is when weak compactness fails to imply strong compactness. We have already encountered problems from the calculus of variations in which we recover the strong compactness of a minimizing sequence from its weak compactness by exploiting the coercivity and the weak lower semi-continuity of the functional undergoing minimization. Here we focus on the case when this compactness issue arises from a concentration phenomena due to an inherent scaling invariance in the problem. The approach we introduce to regain strong convergence (concentration compactness) is to show that concentration only occurs in a small or negligible set. We follow the notes of L. C. Evans [7], but we also refer the reader to P. L. Lions [17, 18]

To illustrate the key points, let us discuss the possibility that a sequence $f_k \rightharpoonup f$ weakly in $L^q(U)$ fails to converge strongly, i.e., $f_k \longrightarrow f$ strongly in $L^q(U)$ does not hold. Indeed, in addition to assuming weak convergence, let us also assume pointwise convergence almost everywhere, $f_k \longrightarrow f$ a.e. in U. This ensures that no wild oscillations may occur, which itself is another potential culprit responsible for the failure of strong convergence. However, even this additional assumption does not guarantee strong convergence due to a possible concentration of mass onto a negligible set. Namely, the obstruction is that the mass $|f_k - f|^q$ may somehow coalesce onto a set with zero Lebesgue measure.

The central example we use to illustrate the concentration compactness principle is the problem on obtaining extremal functions to the sharp Sobolev inequality. In particular, we first give a simple characterization of the non-compactness of the Sobolev embedding in terms of concentration. Then, we use this characterization to recover strong compactness of a minimizing sequence via translations and dilations to obtain an extremal function to the sharp Sobolev inequality. For simplicity, we focus only on the special case where $H^1(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}(\mathbb{R}^n)$, i.e., when $p = 2$ and $q = 2n/(n-2)$ in the Sobolev inequality. Sometimes we denote the borderline Sobolev exponent $2n/(n-2)$ by 2^*. Prior to stating our main results, we review some terminology and basic theorems but we omit their proofs.

Theorem 6.1. *Let $U \subset \mathbb{R}^n$ be a bounded open subset, $1 \leq q < \infty$, and assume $f_k \rightharpoonup f$ in $L^q(U)$. Then*

(a) $\{f_k\}_{k=1}^{\infty}$ is bounded in $L^q(U)$ and

(b) $\|f\|_{L^q(U)} \leq \liminf_{k\to\infty} \|f_k\|_{L^q(U)}$.

(c) (Refinement of Part (b)) If $1 < q < \infty$, $f_k \rightharpoonup f$ in $L^q(U)$ and $\|f\|_{L^q(U)} = \lim_{k\to\infty} \|f_k\|_{L^q(U)}$, then

$$f_k \longrightarrow f \text{ strongly in } L^q(U).$$

Recall the following special case of the Banach-Alaoglu theorem.

Theorem 6.2. *Assume $1 < q < \infty$. If the sequence $\{f_k\}_{k=1}^{\infty}$ is bounded in $L^q(U)$, then it is weakly precompact in $L^q(U)$. That is, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ and a function $f \in L^q(U)$ such that $f_{k_j} \rightharpoonup f$ in $L^q(U)$.*

Remark 6.1. *The previous result holds in the case $q = \infty$ but the convergence of the subsequence in $L^\infty(U)$ is understood in the weak star sense, since $U \subseteq \mathbb{R}^n$ is σ-finite and $L^\infty(U)$ is isometrically isomorphic to the dual space $L^1(U)^*$. Namely, we treat sequences in $L^\infty(U)$ as sequences of bounded linear functionals on $L^1(U)$. The weak compactness in the case $q = 1$ is obviously false. To circumvent this issue, the Riesz Representation Theorem indicates that it is natural to consider $L^1(U)$ as a subset of $\mathcal{M}(U)$, the space of signed finite Radon measures on U.*

Definition 6.1. *A sequence $\{\mu_k\}_{k=1}^{\infty} \subset \mathcal{M}(U)$* **converges weakly** *to $\mu \in \mathcal{M}(U)$, written as*

$$\mu_k \rightharpoonup \mu \text{ in } \mathcal{M}(U),$$

provided that

$$\int_U g\, d\mu_k \longrightarrow \int_U g\, d\mu \text{ as } k \longrightarrow \infty$$

for each $g \in C_c(U)$.

Theorem 6.3. *Assume $\mu_k \rightharpoonup \mu$ weakly in $\mathcal{M}(U)$. Then*

$$\limsup_{k\to\infty} \mu_k(K) \leq \mu(K)$$

for each compact set $K \subset U$, and

$$\mu(V) \leq \liminf_{k\to\infty} \mu_k(V)$$

for each open set $V \subset U$.

Theorem 6.4 (Weak Compactness for Measures)**.** *Assume the sequence $\{\mu_k\}_{k=1}^{\infty}$ is bounded in $\mathcal{M}(U)$. Then there exists a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ and a measure μ in $\mathcal{M}(U)$ such that $\mu_{k_j} \rightharpoonup \mu$ in $\mathcal{M}(U)$.*

Remark 6.2. *We extend the terminology above to the Sobolev space $W^{1,q}(U)$, $1 \leq q < \infty$, by saying that $f_k \rightharpoonup f$ weakly in $W^{1,q}(U)$ whenever $f_k \rightharpoonup f$ in $L^q(U)$ and $Df_k \rightharpoonup Df$ in $L^q(U;\mathbb{R}^n)$.*

Theorem 6.5 (Compactness for Measures)**.** *Assume the sequence $\{\mu_k\}_{k=1}^{\infty}$ is bounded in $\mathcal{M}(U)$. Then $\{\mu_k\}_{k=1}^{\infty}$ is precompact in $W^{-1,q}(U)$ for each $1 \leq q < 1^*$.*

We will need the following refinement of Fatou's lemma (see Lemma A.1) due to Brezis and Lieb.

Theorem 6.6 (Refined Fatou)**.** *Let $1 \leq q < \infty$ and assume $f_k \rightharpoonup f$ weakly in $L^q(U)$ and $f_k \longrightarrow f$ a.e. in U. Then*

$$\lim_{k\to\infty} \left(\|f_k\|^q_{L^q(U)} - \|f_k - f\|^q_{L^q(U)} \right) = \|f\|^q_{L^q(U)}.$$

6.2 Concentration and Sobolev Inequalities

Let C_2 be the best constant in the Gagliardo-Nirenberg-Sobolev inequality in this case (see A.9 in the appendix). There holds the following.

Theorem 6.7. *Assume that $n \geq 3$,*

$$f_k \longrightarrow f \;\; \text{strongly in } \; L^2_{loc}(\mathbb{R}^n), \quad Df_k \rightharpoonup Df \;\; \text{in } \; L^2(\mathbb{R}^n;\mathbb{R}^n).$$

Suppose further that

$$|Df_k|^2 \rightharpoonup \mu \;\; \text{in } \; \mathcal{M}(\mathbb{R}^n), \quad |f_k|^{2^*} \rightharpoonup \nu \;\; \text{in } \; \mathcal{M}(\mathbb{R}^n).$$

(a) Then there exists an at most countable index set J, distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, and non-negative weights $\{\mu_j, \nu_j\}_{j\in J}$ such that

$$\nu = |f|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \geq |Df|^2 + \sum_{j\in J} \mu_j \delta_{x_j}. \tag{6.1}$$

(b) Furthermore,

$$\nu_j \le C_2^{2^*} \mu_j^{2^*/2} \ (j \in J). \tag{6.2}$$

(c) If $f \equiv 0$ and

$$\nu(\mathbb{R}^n)^{1/2^*} \ge C_2 \mu(\mathbb{R}^n)^{1/2},$$

then ν is concentrated at a single point.

Proof. **Step 1:** Assume first that $f \equiv 0$. Choosing $\varphi \in C_c^\infty(\mathbb{R}^n)$, from (A.9) we deduce that

$$\Big(\int_{\mathbb{R}^n} |\varphi f_k|^{2^*} \, dx \Big)^{\frac{1}{2^*}} \le C_2 \Big(\int_{\mathbb{R}^n} |D(\varphi f_k)|^2 \, dx \Big)^{\frac{1}{2}}.$$

Since $f_k \longrightarrow f \equiv 0$ strongly in $L^2_{loc}(\mathbb{R}^n)$, we obtain

$$\Big(\int_{\mathbb{R}^n} |\varphi|^{2^*} \, d\nu \Big)^{\frac{1}{2^*}} \le C_2 \Big(\int_{\mathbb{R}^n} |\varphi|^2 \, d\mu \Big)^{\frac{1}{2}}. \tag{6.3}$$

So by approximation, we have

$$\nu(E)^{1/2^*} \le C_2 \mu(E)^{1/2} \tag{6.4}$$

where $E \subset \mathbb{R}^n$ is any Borel set. Now since μ is a finite measure, the set

$$D := \{ x \in \mathbb{R}^n \,|\, \mu(\{x\}) > 0 \}$$

is at most countable. Thus, we can write $D = \{x_j\}_{j \in J}$, $\mu_j := \mu(\{x_j\})$ $(j \in J)$ so that

$$\mu \ge \sum_{j \in J} \mu_j \delta_{x_j}.$$

From (6.4) and the theory of symmetric derivatives of Radon measures (see Federer), we conclude that $\nu \ll \mu$ and so for each Borel set E,

$$\nu(E) = \int_E D_\mu \nu \, d\mu \tag{6.5}$$

where

$$D_\mu \nu(x) := \lim_{r \to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}. \tag{6.6}$$

But (6.4) implies

$$\frac{\nu(B_r(x))}{\mu(B_r(x))} \le C_2^{2^*} \mu(B_r(x))^{2/(n-2)}, \tag{6.7}$$

provided that $\mu(B_r(x)) \ne 0$. Thus, we infer

$$D_\mu \nu = 0 \quad \mu - a.e. \text{ on } \mathbb{R}^n \backslash D. \tag{6.8}$$

Now define $\nu_j := D_\mu \nu(x_j)\mu_j$. Then (6.5)-(6.8) imply asserts (a) and (b) of the theorem (for $f \equiv 0$).

Step 2: Next, assume the hypotheses of assertion (c) in the theorem. Then (6.4) gives

$$\nu(\mathbb{R}^n)^{1/2^*} = C_2 \mu(\mathbb{R}^n)^{1/2}.$$

Since (6.3) ensures that

$$\Big(\int_{\mathbb{R}^n} |\varphi|^{2^*}\, d\nu\Big)^{\frac{1}{2^*}} \leq C_2 \mu(\mathbb{R}^n)^{\frac{1}{n}} \Big(\int_{\mathbb{R}^n} |\varphi|^2\, d\mu\Big)^{\frac{1}{2}},$$

we deduce that $\nu = C_2^{2^*} \mu(\mathbb{R}^n)^{2/(n-2)}\mu$. Consequently, (6.3) reads

$$\Big(\int_{\mathbb{R}^n} |\varphi|^{2^*}\, d\nu\Big)^{\frac{1}{2^*}} \leq C_2 \nu(\mathbb{R}^n)^{\frac{1}{n}} \Big(\int_{\mathbb{R}^n} |\varphi|^2\, d\nu\Big)^{\frac{1}{2}},$$

and so $\nu(E)^{1/2^*}\nu(\mathbb{R}^n)^{1/n} \leq \nu(E)^{1/2}$ for each Borel set E. This cannot happen if ν is concentrated at more than one point.

Step 3: Now assume $f \not\equiv 0$ and write $g_k := f_k - f$. The calculations in the Steps 1 and 2 apply to $\{g_k\}_{k=1}^\infty$ as well. Moreover, there holds

$$|Dg_k|^2 = |Df_k|^2 - 2Df_k \cdot Df + |Df|^2 \rightharpoonup \mu - |Df|^2 \text{ in } \mathcal{M}(\mathbb{R}^n),$$

and Theorem 6.6 implies $|g_k|^{2^*} \rightharpoonup \nu - |f|^{2^*}$ in $\mathcal{M}(\mathbb{R}^n)$. This completes the proof. and

$\square$

6.3 Minimizers for Critical Sobolev Inequalities

Let $n \geq 3$ and consider the problem of minimizing the functional

$$I[w] = \int_{\mathbb{R}^n} |Dw|^2\, dx, \tag{6.9}$$

over the admissible set

$$M := \{w \in L^{2^*}(\mathbb{R}^n) \,|\, \|w\|_{L^{2^*}(\mathbb{R}^n)} = 1,\ Dw \in L^2(\mathbb{R}^n;\mathbb{R}^n)\}.$$

Notice carefully that

$$I := \inf_{w \in M} I[w] = C_2^{-2}.$$

Our goal is to show that this infimum is indeed obtained by a suitable minimizer. On a related note, we may also consider the same minimization problem but on an arbitrary domain U with functional

$$I_U[w] = \int_U |Dw|^2\, dx$$

undergoing minimization over

$$M_U := \{w \in L^{2^*}(U) \,|\, \|w\|_{L^{2^*}(U)} = 1,\ Dw \in L^2(U;\mathbb{R}^n)\}.$$

Interestingly enough, the infimum here is also given by the best constant in the Gagliardo-Nirenberg-Sobolev inequality, i.e.,

$$\min_{w\in M_U} I_U[w] = I = C_2^{-2},$$

but the minimum is not achieved for $U \neq \mathbb{R}^n$. In other words, the best constant in the sharp Sobolev inequality does not depend on the domain and the culprit responsible for this is the scaling invariance

$$u(x) \mapsto u_R(x) := R^{n/2^*}u(Rx) = R^{(n-2)/2}u(Rx),\ \ R > 0,$$

with respect to the norms in the Sobolev inequality. In particular, if for example, $u \in H^1(\mathbb{R}^n)$ with unit norm, then $\|u_R\|_{L^{2^*}(U)} = \|u\|_{L^{2^*}(U)} = 1$ but $u_R \rightharpoonup 0$ in $H^1(\mathbb{R}^n)$ as $R \longrightarrow \infty$. Therefore, relative compactness of minimizing sequences is not expected to hold. What ultimately saves us is the actions of rescaling and translations, which can recover the relative compactness of minimizing sequences.

Remark 6.3. *(a) Recall that the method of moving planes indicates that the critical points of the functional I, which includes its minimizers, are essentially unique. Namely, all critical points must admit the form*

$$u_{\varepsilon,x_0}(x) = c(n)\Big(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2}\Big)^{\frac{n-2}{2}} \tag{6.10}$$

for some $\varepsilon > 0$ and some point $x_0 \in \mathbb{R}^n$.

(b) The classification of critical points in (a) also illustrates the concentration property which occurs in the critical Sobolev inequality. Indeed, upon normalizing, there holds

$$C\|u_{\varepsilon,x_0}\|_{H^1(\mathbb{R}^n)} = \|u_{\varepsilon,x_0}\|_{L^{2^*}(\mathbb{R}^n)} = 1$$

so that the sequence $\{u_{\varepsilon,x_0}\}_{\varepsilon>0}$ is bounded in these norms. However, as $\varepsilon \longrightarrow 0$, we have that

$$\begin{cases} u_{\varepsilon,x_0}(x) \longrightarrow 0 & \textit{for } x \neq x_0, \\ u_{\varepsilon,x_0}(x) = 1/\varepsilon^{(n-2)/2} \longrightarrow \infty & \textit{for } x = x_0. \end{cases}$$

Now we choose a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset M$ with

$$I[u_k] \longrightarrow I. \tag{6.11}$$

We may assume $Du_k \rightharpoonup Du$ in $L^2(\mathbb{R}^n;\mathbb{R}^n)$ and $u_k \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^n)$. Recall from Chapter 2 that

$$I[u] \le \liminf_{k\to\infty} I[u_k] = \inf_{w\in M} I[w].$$

Hence, u is a minimizer as long as $u \in M$. Now, since we have

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \le 1, \tag{6.12}$$

what is only left to verify is if $\|u\|_{L^{2^*}(\mathbb{R}^n)} = 1$. Once we verify this, we are done. Before we state and prove the main result, for $v \in M$, $y \in \mathbb{R}^n$ and $s > 0$, we define the rescaled function

$$v^{y,s}(x) := s^{-\frac{n-2}{2}} v\Big(\frac{x-y}{s}\Big) \ \ (x \in \mathbb{R}^n).$$

Theorem 6.8. *Let $\{u_k\}_{k=1}^\infty \subset M$ satisfy* (6.11). *Then there exist translations $\{y_k\}_{k=1}^\infty \subset \mathbb{R}^n$ and dilations $\{s_k\}_{k=1}^\infty \subset (0,\infty)$ such that the rescaled family $\{u_k^{y_k,s_k}\}_{k=1}^\infty \subset M$ is strongly precompact in $L^{2^*}(\mathbb{R}^n)$. In particular there exists a minimizer $u \in M$ of the functional I.*

Sketch of Proof. We outline the proof in five main steps.
Step 1: Define the Lévy concentration functions

$$Q_k(t) := \sup_{y\in\mathbb{R}^n} \int_{B_t(y)} |u_k|^{2^*}\,dx \ \ (t > 0, k = 1,2,3,\ldots).$$

Then $Q_k^{y,s}(t) = Q_k^{y,1}(t/s)$ where $Q_k^{y,s}$ is the concentration function of $u_k^{y,s}$. The fact that

$$\lim_{t\to\infty} Q_k(t) = 1$$

ensures we can choose dilations $\{s_k\}_{k=1}^\infty$ such that

$$Q_k^{y,s_k}(1) = 1/2 \ \text{ for all } y \in \mathbb{R}^n,\ k = 1,2,3,\ldots.$$

Then this allows us to select translations $\{y_k\}_{k=1}^\infty$ so that the measures, $\nu_k^{y_k,s_k} = |u_k^{y_k,s_k}|^{2^*}$ $(k = 1,2,3\ldots)$, are tight in $\mathcal{M}(\mathbb{R}^n)$.

Step 2: To simplify notation, we assume the dilations and translations of step one were unnecessary and so $Q_k(1) = 1/2$ $(k = 1,2,3,\ldots)$ and the measures $\{\nu_k\}_{k=1}^\infty$ are tight. Thus, passing to a subsequence, if necessary, we may assume

$$\nu_k \rightharpoonup \nu \ \text{ in } \mathcal{M}(\mathbb{R}^n), \quad \nu(\mathbb{R}^n) = 1. \tag{6.13}$$

We may also assume that

$$\mu_k \rightharpoonup \mu \ \text{ in } \mathcal{M}(\mathbb{R}^n) \tag{6.14}$$

for $\mu_k := |Du_k|^2$ $(k = 1,2,3,\ldots)$.

Step 3: We claim that $u \not\equiv 0$.

Assume the contrary. By noting that $\mu_k(\mathbb{R}^n) \longrightarrow I$, $\mu(\mathbb{R}^n) \le I = C_2^{-2}$, and (6.13), we use part (c) of Theorem 6.7 to get that ν is concentrated at a single point $x_0 \in \mathbb{R}^n$. From this we deduce the contradiction

$$\frac{1}{2} = Q_k(1) \ge \int_{B_1(x_0)} |u_k|^{2^*}\, dx \longrightarrow 1.$$

Step 4: We claim that $u \in M$.

Assume otherwise, i.e., assume that $\|u\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = \lambda \in (0,1)$. Setting

$$M_\lambda := \{w \in L^{2^*}(\mathbb{R}^n) \mid \|w\|_{L^{2^*}(\mathbb{R}^n)} = \lambda,\ Dw \in L^2(\mathbb{R}^n;\mathbb{R}^n)\},$$

we write

$$I_\lambda := \inf_{w\in M_\lambda} I[w].$$

Then $I_\lambda = \lambda^{2/2^*} I$.

Step 5: According to (a) and (b) of Theorem 6.7, we have

$$\nu = |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \mu \ge |Du|^2 + \sum_{j\in J} \mu_j \delta_{x_j}$$

for some countable set of points $\{x_j\}_{j\in J}$ and positive weights $\{\mu_j, \nu_j\}_{j\in J}$, satisfying

$$\lambda + \sum_{j\in J} \nu_j = 1, \qquad \mu_j \ge \nu_j^{2/2^*} I \ (j \in J).$$

Hence, we arrive at the contradiction

$$\begin{aligned} I \ge \mu(\mathbb{R}^n) &\ge \int_{\mathbb{R}^n} |Du|^2\, dx + \sum_{j\in J} \mu_j \\ &\ge I_\lambda + \sum_{j\in J} \mu_j \ge \Big(\lambda^{2/2^*} + \sum_{j\in J} \nu_j^{2/2^*}\Big) I \\ &> I, \end{aligned}$$

and this completes the proof. □

Remark 6.4. *Roughly speaking, Steps 3 to 5 in the proof show that vanishing and dichotomy in the principle of concentration compactness do not occur and therefore, compactness must hold (see Proposition 2.1). Step 5, in particular, shows that if a portion of the mass concentrates, our minimization problem splits into two parts, the sum of whose energies strictly exceeds the energy were splitting not to occur.*

APPENDIX A

Basic Inequalities, Embeddings and Convergence Theorems

This appendix covers some basic inequalities, embeddings and convergence results that we frequently apply throughout.

A.1 Basic Inequalities

Theorem A.1 (Cauchy's inequality)**.** *There holds for $a, b \in \mathbb{R}$,*

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

More generally, we have Cauchy's inequality with ϵ:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \quad (a, b > 0,\, \epsilon > 0).$$

Theorem A.2 (Young's inequality)**.** *Let $1 < p, q < \infty$ and $1/p + 1/q = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem A.3 (Jensen's inequality)**.** *Assume $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is convex, i.e.,*

$$f(\tau x + (1 - \tau)y) \leq \tau f(x) + (1 - \tau)f(y)$$

for all $x, y \in \mathbb{R}^m$, and $U \subset \mathbb{R}^n$ is bounded and open. Let $\mathbf{u} : U \longrightarrow \mathbb{R}^m$ be summable. Then

$$f\Big(\frac{1}{|U|}\int_U \mathbf{u}\, dx\Big) \leq \frac{1}{|U|}\int_U f(\mathbf{u})\, dx.$$

Theorem A.4 (Hölder's inequality)**.** *Assume* $1 \le p, q \le \infty$ *and* $1/p+1/q = 1$*. If* $u \in L^p(U)$*,* $v \in L^q(U)$*, then*

$$\int_U |uv|\, dx \le \|u\|_{L^p(U)} \|v\|_{L^q(U)}. \tag{A.1}$$

Proof. Let $u \in L^p(U)$ and $v \in L^q(U)$. From the homogeneity of the L^p norms, we can assume that $\|u\|_{L^p(U)} = \|v\|_{L^q(U)} = 1$. Then by Young's inequality of Theorem A.2,

$$\int_U uv\, dx \le \frac{1}{p} \int_U u^p\, dx + \frac{1}{q} \int_U v^q\, dx = \frac{1}{p} + \frac{1}{q} = 1 = \|u\|_{L^p(U)} \|v\|_{L^q(U)}.$$

□

An easy extension of this inequality is the following whose proof we omit.

Theorem A.5 (General Hölder's Inequality)**.** *Let* $1 \le p_1, p_2, \ldots, p_m \le \infty$ *with* $\sum_{k=1}^m \frac{1}{p_k} = 1$*, and assume* $u_k \in L^{p_k}(U)$ *for* $k = 1, \ldots, m$*. Then*

$$\int_U |u_1 \ldots u_m|\, dx \le \prod_{k=1}^m \|u_k\|_{L^{p_k}(U)}. \tag{A.2}$$

Theorem A.6 (L^p interpolation)**.** *Assume that* $1 \le p \le r \le q \le \infty$ *and*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{(1-\theta)}{q}.$$

Suppose also that $u \in L^p(U) \cap L^q(U)$*. Then* $u \in L^r(U)$ *and*

$$\|u\|_{L^r(U)} \le \|u\|^{\theta}_{L^p(U)} \|u\|^{1-\theta}_{L^q(U)}. \tag{A.3}$$

Proof. Since $\frac{\theta r}{p} + \frac{(1-\theta)r}{q} = 1$, Hölder's inequality yields

$$\int_U |u|^r\, dx = \int_U |u|^{\theta r} |u|^{(1-\theta)r}\, dx \le \left(\int_U |u|^{\theta r \frac{p}{\theta r}}\right)^{\frac{\theta r}{p}} \left(\int_U |u|^{(1-\theta)r \frac{q}{(1-\theta)r}}\, dx\right)^{\frac{(1-\theta)r}{q}}.$$

□

Theorem A.7 (Gronwall's inequality)**.** *Let* $\eta(\cdot)$ *be a non-negative absolutely continuous (i.e., differentiable a.e.) function on* $[0, T]$*, which satisfies for a.e.* t*, the differential inequality*

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t), \tag{A.4}$$

where $\phi(t)$ *and* $\psi(t)$ *are non-negative, summable functions on* $[0, T]$*. Then*

$$\eta(t) \le e^{\int_0^t \phi(s)\, ds} \Big(\eta(0) + \int_0^t \psi(s)\, ds\Big) \quad \text{for all } 0 \le t < T. \tag{A.5}$$

In particular, if $\eta(0) = 0$ *and*

$$\eta'(t) \le \phi\eta \quad \text{on } [0, T],$$

then

$$\eta \equiv 0 \quad \text{on } [0, T].$$

Proof. From (A.4),

$$\frac{d}{ds}\Big(\eta(s)e^{-\int_0^s \phi(r)\,dr}\Big) = e^{-\int_0^s \phi(r)\,dr}(\eta'(s) - \phi(s)\eta(s)) \le e^{-\int_0^s \phi(r)\,dr}\psi(s) \text{ for a.e. } 0 \le s \le T.$$

Integrating this we get, for each $0 \le t \le T$,

$$\eta(t)e^{-\int_0^t \phi(r)\,dr} \le \eta(0) + \int_0^t e^{-\int_0^s \phi(r)\,dr}\psi(s)\,ds \le \eta(0) + \int_0^t \psi(s)\,ds.$$

□

Sometimes, it is more convenient to use the integral form of Gronwall's inequality.

Theorem A.8. *Let $\xi(t)$ be a non-negative, summable function on $[0,T]$ which satisfies, for a.e. t the integral inequality*

$$\xi(t) \le C_1 \int_0^t \xi(s)\,ds + C2, \tag{A.6}$$

for some constants $C_1, C_2 \ge 0$. Then

$$\xi(t) \le C_2(1 + C_1te^{C_1t}) \text{ for a.e. } 0 \le t \le T.$$

In particular, if

$$\xi(t) \le C1 \int_0^t \xi(s)\,ds \text{ for a.e. } 0 \le t \le T,$$

then

$$\xi \equiv 0 \text{ on } [0,T].$$

Proof. Set $\eta(t) = \int_0^t \xi(s)\,ds$ so that $\eta'(t) \le C_1\eta(t) + C_2$ for a.e. t in $[0,T]$. According to the differential version of Gronwall's inequality, we obtain

$$\eta(t) \le e^{C_1t}(\eta(0) + C_2t) = C_2te^{C_1t}.$$

The result then follows from (A.6) since

$$\xi(t) \le C_1\eta(t) + C_2 \le C_2(1 + C_1te^{C_1t}).$$

□

A.2 Sobolev Inequalities

Next, we introduce and prove the Gagliardo–Nirenberg–Sobolev inequality and Morrey's inequality. For each estimate, we establish its corresponding Sobolev embedding theorems.

Theorem A.9 (Gagliardo-Nirenberg-Sobolev)**.** *Assume* $1 \le p < n$ *and denote* $p^* := np/(n-p)$*. There exists a constant* $C = C(n,p)$ *such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C(n,p)\|Du\|_{L^p(\mathbb{R}^n)} \tag{A.7}$$

for all $u \in C_c^1(\mathbb{R}^n)$*.*

Remark A.1. *Note that the functions* u *must have compact support to discriminate from obvious cases such as constant functions. However, it is interesting that the constant* C *does not depend on the size of the support of* u*.*

Proof. **Step 1:** We first prove the estimate for $p = 1$.

Since u has compact support, for each $i = 1, 2, \dots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i;$$

and so for $i = 1, 2, \dots, n$,

$$|u(x)| \le \int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)|\, dy_i.$$

Therefore,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \dots, y_i, \dots, x_n)|\, dy_i \right)^{\frac{1}{n-1}}.$$

Integrating this inequality with respect to x_1 yields

$$\begin{aligned}
\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&= \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&\le \left(\int_{-\infty}^{\infty} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dy_i \right)^{\frac{1}{n-1}},
\end{aligned} \tag{A.8}$$

where we used the general Hölder's inequality in the last inequality. Now integrate (A.8) with respect to x_2:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\, dx_1\, dx_2 \le \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|\, dx_1\, dx_2 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1, i\ne 2}^{n} I_i^{\frac{1}{n-1}}\, dx_2,$$

where
$$I_1 = \int_{-\infty}^{\infty} |Du|\,dy_1, \quad I_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\,dx_1\,dy_i \quad (i = 3, 4, \ldots, n).$$
Applying the general Hölder's inequality once more to this yields
$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\,dx_1\,dx_2 \le{}& \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\,dx_1\,dx_2\right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\,dy_1\,dx_2\right)^{\frac{1}{n-1}}\\ &\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|\,dx_1\,dx_2\,dy_i\right)^{\frac{1}{n-1}}\end{aligned}$$
We continue integrating with respect to $x_3, x_4, \ldots, x_n$, until we arrive at
$$\begin{aligned}\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx &\le \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |Du|\,dx_1 \ldots dy_i, \ldots dx_n\right)^{\frac{1}{n-1}}\\ &= \left(\int_{\mathbb{R}^n} |Du|\,dx\right)^{\frac{n}{n-1}}. \end{aligned} \tag{A.9}$$
Hence, this proves the theorem for $p = 1$.
Step 2: Consider the case where $p \in (1, n)$. If we apply estimate (A.9) to $v := |u|^\gamma$ ($\gamma > 1$ is to be determined below), we obtain
$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n} |Dv|\,dx = \gamma\int_{\mathbb{R}^n} |u|^{\gamma-1}|Du|\,dx\\ &\le \gamma\left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p\,dx\right)^{\frac{1}{p}}. \end{aligned} \tag{A.10}$$
Set
$$\gamma = \frac{p(n-1)}{n-p} > 1$$
so that
$$\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*.$$
Thus, (A.10) becomes
$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}} \le C\left(\int_{\mathbb{R}^n} |Du|^p\,dx\right)^{\frac{1}{p}}$$
and this completes the proof. □

Theorem A.10 (Morrey's inequality)**.** *Assume $n < p \le \infty$. Then there exists a constant $C(n,p)$ such that*
$$\|u\|_{C^{0,1-n/p}(\mathbb{R}^n)} \le C(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{A.11}$$
for all $u \in C^1(\mathbb{R}^n)$.

Proof. **Step 1:** We claim there exists a constant $C = C(n)$ depending only on n such that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y)-u(x)|\,dy \le C\int_{B_r(x)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy \tag{A.12}$$

for each open ball $B_r(x) \subset \mathbb{R}^n$.

To show this, fix any point $w \in \partial B_1(0)$. Then, if $0 < s < r$,

$$\begin{aligned}|u(x+sw)-u(x)| &= \left|\int_0^s \frac{d}{dt}u(x+tw)\,dt\right| = \left|\int_0^s Du(x+tw)\cdot w\,dt\right| \\ &\le \int_0^s |Du(x+tw)|\,dt.\end{aligned}$$

Hence,

$$\int_{\partial B_1(0)} |u(x+sw)-u(x)|\,ds_w \le \int_0^s\int_{\partial B_1(0)} |Du(x+tw)|\,ds_w\,dt. \tag{A.13}$$

We estimate the right-hand side of this inequality to get

$$\begin{aligned}\int_0^s\int_{\partial B_1(0)} |Du(x+tw)|\,ds_w\,dt &= \int_0^s\int_{\partial B_t(x)} \frac{|Du(y)|}{t^{n-1}}\,ds_y\,dt \\ &= \int_{B_s(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy \le \int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy,\end{aligned}$$

where $y = x + tw$ and $t = |x-y|$. The left-hand side can be written

$$\int_{\partial B_1(0)} |u(x+sw)-u(x)|\,ds_w = \frac{1}{s^{n-1}}\int_{\partial B_s(x)} |u(z)-u(x)|\,ds_z,$$

where $z = x + sw$. Combining the preceding two calculations in (A.13), we obtain the estimate

$$\int_{\partial B_s(x)} |u(z)-u(x)|\,ds_z \le s^{n-1}\int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy.$$

Integrate this with respect to s from 0 to r yields

$$\int_{B_r(x)} |u(y)-u(x)|\,dy \le \frac{r^n}{n}\int_{B_r(x)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy.$$

This proves our first claim.

Step 2: Fix $x \in \mathbb{R}^n$. Applying estimate (A.12) then Hölder's inequality, we get

$$\begin{aligned}|u(x)| &\le \frac{1}{|B_1(x)|}\int_{B_1(x)} |u(x)-u(y)|\,dy + \frac{1}{|B_1(x)|}\int_{B_1(x)} |u(y)|\,dy \\ &\le C\int_{B_r(x)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy + C\|u\|_{L^p(B_1(x))} \\ &\le C\left(\int_{\mathbb{R}^n} |Du|^p\,dy\right)^{\frac{1}{p}}\left(\int_{B_1(x)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}\,dy\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^n)} \\ &\le C\|u\|_{W^{1,p}(\mathbb{R}^n)}.\end{aligned}$$

The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$, so that

$$\int_{B_1(x)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}\, dy < \infty.$$

As $x \in \mathbb{R}^n$ is arbitrary, there holds

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Step 3: Next, choose any two points $x, y \in \mathbb{R}^n$ and set $r := |x-y|$. Let $W := B_r(x) \cap B_r(y)$. Then

$$|u(x) - u(y)| \leq \frac{1}{|W|}\int_W |u(x) - u(z)|\, dz + \frac{1}{|W|}\int_W |u(y) - u(z)|\, dz = I_1 + I_2.$$

Furthermore, estimate (A.12) allows us to estimate

$$\begin{aligned}
I_1 = \frac{1}{|W|}\int_W |u(x) - u(z)|\, dz &\leq C\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(x) - u(z)|\, dz\right) \\
&\leq C\left(\int_{B_r(x)} |Du|^p\, dz\right)^{\frac{1}{p}} \left(\int_{B_r(x)} \frac{dz}{|x-z|^{(n-1)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}} \\
&\leq C\left(r^{n-(n-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)} \\
&\leq Cr^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, we calculate

$$I_2 = \frac{1}{|W|}\int_W |u(y) - u(z)|\, dz \leq Cr^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)}$$

Hence,

$$|u(x) - u(y)| \leq Cr^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)} = C|x-y|^{1-\frac{n}{p}}\|Du\|_{L^p(\mathbb{R}^n)},$$

therefore,

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C\|Du\|_{L^p(\mathbb{R}^n)}.$$

□

A.2.1 Extension and Trace Operators

Although we use the Gagliardo-Nirenberg-Sobolev and Morrey inequalities to prove the classical Sobolev embedding theorems, we shall also make use of the following basic results.

Theorem A.11 (Extension Theorem). *Assume U is bounded and ∂U is C^1. Select a bounded open set V such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$ there hold

(a) $Eu = u$ a.e. in U,

(b) Eu has support within V,

(c) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$

with the positive constant $C = C(p, U, V)$ depending only on p, U and V. Here Eu is called an extension of u to $\mathbb{R}^n$.

Theorem A.12 (Trace Theorem). *Assume U is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \longrightarrow L^p(\partial U)$$

such that

(a) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$,

(b) $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for each $u \in W^{1,p}(U)$

with the positive constant $C = C(p, U)$ depending only on p and U.

Remark A.2. *The trace operator T enables us to assign boundary values along ∂U to functions in $W^{1,p}(U)$. This is especially useful for studying the Dirichlet problem and characterizing the space $W_0^{1,p}(U)$, the closure of $C_c^\infty(U)$ in $W^{1,p}(U)$, as the $W^{1,p}$ functions vanishing at the boundary. For example, if U is bounded and ∂U is C^1, and $u \in W^{1,p}(U)$, then (see [8][Theorem 2 on page 273])*

$$u \in W_0^{1,p}(U) \text{ if and only if } Tu = 0 \text{ on } \partial U.$$

The next property concerns the global approximation of functions in $W^{1,p}(U)$ by smooth functions.

Theorem A.13 (Density Theorem). *Assume that U is bounded and suppose that $u \in W^{1,p}(U)$ for some $1 \leq p < \infty$.*

(a) There exists functions $u_m \in C^\infty(U) \cap W^{1,p}(U)$ such that

$$u_m \longrightarrow u \text{ in } W^{1,p}(U).$$

(b) If, in addition, ∂U is C^1, then statement (a) holds but the approximating sequence of functions can be taken to be smooth up to the boundary, i.e., $u_m \in C^\infty(\bar{U})$.

A.2.2 Sobolev Embeddings and Poincaré Inequalities

The first embedding theorem follows from the Gagliardo-Nirenberg-Sobolev inequality.

Theorem A.14 (Sobolev embedding 1)**.** *Let U be a bounded open subset of $\mathbb{R}^n$ and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$ with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C(n,p,U)\|u\|_{W^{1,p}(U)},$$

where the constant $C = C(n,p,U)$ depends only on $n, p,$ and U.

Proof. Since ∂U is C^1, the extension theorem of Theorem A.11 implies that there exists an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\bar{u} = u$ in U, $\bar{u}$ has compact support, and

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \tag{A.14}$$

Since $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ has compact support, the Density theorem or Theorem A.13 implies that there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ $(m = 1, 2, \ldots)$ such that

$$u_m \longrightarrow \bar{u} \ \text{ in } \ W^{1,p}(\mathbb{R}^n). \tag{A.15}$$

From the Gagliardo–Nirenberg–Sobolev inequality, we obtain

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}$$

for all $l, m \geq 1$. Hence,

$$u_m \longrightarrow \bar{u} \ \text{ in } \ L^{p^*}(\mathbb{R}^n). \tag{A.16}$$

Moreover, the Gagliardo–Nirenberg–Sobolev inequality also implies

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)},$$

Therefore, (A.15) and (A.16) imply

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)},$$

This inequality and (A.14) complete the proof. □

Theorem A.15 (Sobolev embedding 2)**.** *Assume U is a bounded open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \leq C(n,p,q,U)\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, where the constant $C = C(n,p,q,U)$ depends only on $n, p, q,$ and U. In particular, for all $1 \leq p \leq \infty$,

$$\|u\|_{L^p(U)} \leq C(n,p,q,U)\|Du\|_{L^p(U)}. \tag{A.17}$$

Remark A.3. *Estimate* (A.17) *is sometimes called Poincaré's inequality. Consequently, this inequality implies the norm* $\|Du\|_{L^p(U)}$ *is equivalent to* $\|u\|_{W^{1,p}(U)}$ *in* $W_0^{1,p}(U)$ *provided* U *is bounded.*

Proof of Theorem A.15. Since $u \in W_0^{1,p}(U)$, there exist functions $u_m \in C_c^\infty(U)$ $(m = 1, 2, \ldots)$ converging to u in $W^{1,p}(U)$. We extend each function u_m to be 0 on $\mathbb{R}^n \backslash \bar{U}$ (we do not need to invoke the extension theorem) and apply the Gagliardo–Nirenberg–Sobolev inequality to obtain

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}.$$

Since $\mu(U) < \infty$, basic interpolation theory says the identity map, $I : L^{p^*}(U) \longrightarrow L^q(U)$, is bounded provided $1 \leq q \leq p^*$, i.e., $\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}(U)}$ if $1 \leq q \leq p^*$. $\square$

Definition A.1. *We say* u^* *is a version of a given function* u *if* $u = u^*$ *a.e.*

The next embedding theorem is a result of Morrey's inequality.

Theorem A.16 (Sobolev embedding 3)**.** *Let* U *be a bounded open subset of* $\mathbb{R}^n$ *and suppose its boundary* ∂U *is* C^1. *Assume* $n < p \leq \infty$ *and* $u \in W^{1,p}(U)$. *Then* u *has a version* $u^* \in C^{0,\gamma}(\bar{U})$, *for* $\gamma = 1 - \frac{n}{p}$, *with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C(n, p, U)\|u\|_{W^{1,p}(U)}.$$

The constant $C = C(n, p, U)$ *depends only on* n, p *and* U*.*

Proof. We only consider the case $n < p < \infty$ since the case $p = \infty$ is easy to prove directly. Since ∂U is C^1, the extension theorem implies that there is an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\bar{u} = u$ in U, $\bar{u}$ has compact support, and

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}. \tag{A.18}$$

Since $\bar{u}$ has compact support, Theorem A.13 implies there exist functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \longrightarrow \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n). \tag{A.19}$$

According to Morrey's inequality, $\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$ where $\gamma = 1 - \frac{n}{p}$ for all $l, m \geq 1$. Hence, there exists a function $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ such that

$$u_m \longrightarrow u^* \text{ in } C^{0,\gamma}(\mathbb{R}^n). \tag{A.20}$$

Owing to (A.19) and (A.20), we see that $u = u^*$ a.e. in U, so u^* is a version of u. Morrey's inequality also implies $\|u_m\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$. Thus, (A.19) and (A.20) imply

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}.$$

This inequality and (A.18) complete the proof of the theorem. $\square$

The previous Sobolev inequalities for $W^{1,p}(U)$ can be further generalized to the Sobolev spaces $W^{k,p}(U)$ for $k \in \mathbb{N}$.

Theorem A.17 (General Sobolev inequalities). *Let U be a bounded open subset of $\mathbb{R}^n$ with a C^1 boundary ∂U. Assume $u \in W^{k,p}(U)$.*

(i) If $k < \frac{n}{p}$, then $u \in L^q(U)$ where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n} \Longleftrightarrow q = \frac{np}{n-kp}.$$

We have, in addition, the estimate

$$\|u\|_{L^q(U)} \leq C(k,n,p,U)\|u\|_{W^{1,p}(U)}.$$

The constant $C = C(k,n,p,U)$ depends only on k, n, p, and U.

(ii) If $k > \frac{n}{p}$, then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & if\ \frac{n}{p} \text{ is not an integer },\\ \text{any positive number } < 1, & if\ \frac{n}{p} \text{ is an integer.} \end{cases}$$

We have, in addition, the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})} \leq C(k,n,p,\gamma,U)\|u\|_{W^{k,p}(U)},$$

the constant $C = C(k,n,p,\gamma,U)$ depending only on k, n, p, γ, and U.

Proof. The proof is standard, similar to the aforementioned special cases above, and we refer the reader to Evans [8] for more details. □

Remark A.4 (Case $p = n$). *In the endpoint borderline case for $p \in [1,n)$, $p^* = np/(n-p) \longrightarrow +\infty$ by sending $p \longrightarrow n$ which suggests that $W^{1,n}(U) \subset L^\infty(U)$. Unfortunately, this only holds when $n = 1$ and fails for $n \geq 2$. For example, if we take $n \geq 2$ and $U = B_1(0) \subset \mathbb{R}^n$, then the function $\log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$ but not to $L^\infty(U)$. However, $BMO(U)$, the space of functions with bounded mean oscillation, is the proper embedding space to replace $L^\infty(U)$ in order to preserve the embedding of the Sobolev space (see Corollary A.1).*

The next theorem is on the compact embedding of Sobolev spaces into Lebesgue spaces.

Theorem A.18 (Rellich–Kondrachov compactness). *Assume U is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary ∂U. Suppose $1 \leq p < n$, then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \leq q < p^*$.

Proof. **1.** Fix $1 \leq q < p^*$ and note that since U is bounded, Theorem A.14 implies $W^{1,p}(U) \subset L^q(U)$ and $\|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}$. Thus, it remains to show that if $\{u_m\}_{m=1}^\infty$ is a bounded sequence in $W^{1,p}(U)$, there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ which converges in $L^q(U)$.
2. By the Extension theorem, we may assume, without loss of generality, that $U = \mathbb{R}^n$ and the functions $\{u_m\}_{m=1}^\infty$ all have compact support in some bounded open set $V \subset \mathbb{R}^n$. We also may assume

$$\sup_m \|u_m\|_{W^{1,p}(U)} < \infty. \tag{A.21}$$

3. We first examine the smoothed functions

$$u_m^\epsilon := \eta_\epsilon * u_m \ (\epsilon > 0, m = 1, 2, 3, \ldots),$$

where η_ϵ denotes the standard mollifier. We may assume that the functions $\{u_m^\epsilon\}_{m=1}^\infty$ all have support in V as well.
4. We claim that

$$u_m^\epsilon \longrightarrow u_m \text{ in } L^q(V) \text{ as } \epsilon \longrightarrow 0 \text{ uniformly in } m. \tag{A.22}$$

To prove this, we note that if u_m is smooth, then

$$\begin{aligned}
u_m^\epsilon(x) - u_m(x) &= \frac{1}{\epsilon^n}\int_{B_\epsilon(x)} \eta\left(\frac{x-z}{\epsilon}\right)(u_m(z) - u_m(x))\,dz \\
&= \int_{B_1(0)} \eta(y)(u_m(x-\epsilon y) - u_m(x))\,dy \\
&= \int_{B_1(0)} \eta(y)\int_0^1 \frac{d}{dt}(u_m(x-\epsilon t y))\,dt\,dy \\
&= -\epsilon\int_{B_1(0)} \eta(y)\int_0^1 Du_m(x-\epsilon t y)\cdot y\,dt\,dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_V |u_m^\epsilon(x) - u_m(x)|\,dx &\leq \epsilon\int_{B_1(0)} \eta(y)\int_0^1\int_V |Du_m(x-\epsilon t y)|\,dx\,dt\,dy \\
&\leq \epsilon\int_V |Du_m(z)|\,dz.
\end{aligned}$$

By approximation, this estimate holds if $u_m \in W^{1,p}(V)$. Since V is bounded, we obtain

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon\|Du_m\|_{L^1(V)} \leq \epsilon C\|Du_m\|_{L^p(V)},$$

By virtue of (A.21), we have

$$u_m^\epsilon \longrightarrow u_m \text{ in } L^1(V) \text{ uniformly in } m. \tag{A.23}$$

Then since $1 \le q < p^*$, the L^p interpolation inequality yields

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta},$$

where $\frac{1}{q} = \theta + \frac{(1-\theta)}{p^*}$ and $\theta \in (0,1)$. As a consequence of (A.21) and the Gagliardo–Nirenberg–Sobolev inequality, we obtain

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le C\|u_m^\epsilon - u_m\|_{L^1(V)}^\theta.$$

Hence, (A.22) follows from (A.21).
5. Next, we claim that for each $\epsilon > 0$, the sequence $\{u_m\}_{m=1}^\infty$ is uniformly bounded and equicontinuous.

Indeed, if $x \in \mathbb{R}^n$, then

$$|u_m^\epsilon(x)| \le \int_{B_\epsilon(x)} \eta_\epsilon(x-y)|u_m(y)|\,dy \le \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \le C\epsilon^{-n} < \infty,$$

for $m = 1, 2, \ldots$. Similarly,

$$|Du_m^\epsilon(x)| \le \int_{B_\epsilon(x)} |D\eta_\epsilon(x-y)||u_m(y)|\,dy \le \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \le C\epsilon^{-(n+1)} < \infty,$$

for $m = 1, 2, \ldots$. Thus, the claim follows from these two estimates.
6. Now fix $\delta > 0$. We show that there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty \subset \{u_m\}_{m=1}^\infty$ such that

$$\lim_{j,k\longrightarrow\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta. \tag{A.24}$$

To see this, we employ (A.22) to select $\epsilon > 0$ suitably small such that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le \delta/2 \tag{A.25}$$

for $m = 1, 2, \ldots$.

Now observe that since the functions $\{u_m\}_{m=1}^\infty$, and thus the functions $\{u_m^\epsilon\}_{m=1}^\infty$, have support in some fixed bounded set $V \subset \mathbb{R}^n$, we can apply the claim in 5. and the Arzelà–Ascoli compactness theorem to extract a subsequence $\{u_{m_j}^\epsilon\}_{j=1}^\infty \subset \{u_m^\epsilon\}_{m=1}^\infty$ which converges uniformly on V. Therefore,

$$\limsup_{j,k\longrightarrow\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0.$$

But then this combined with (A.25) imply

$$\limsup_{j,k\longrightarrow\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta.$$

This proves (A.24).

7. By applying assertion (A.24) with $\delta = 1, 1/2, 1/3, \ldots$ and use a standard diagonal argument to extract a subsequence $\{u_{m_j}\}_{j=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ satisfying

$$\limsup_{l,k \longrightarrow \infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

This completes the proof of the theorem. $\square$

Remark A.5. *Since $p^* > p$ and $p^* \longrightarrow \infty$ as $p \longrightarrow n$, we have*

$$W^{1,p}(U) \subset\subset L^p(U)$$

for all $1 \leq p \leq \infty$. In addition, note that

$$W_0^{1,p}(U) \subset\subset L^p(U),$$

even if we do not assume ∂U is C^1.

The Rellich–Kondrachov compactness theorem allows us to establish the following Poincaré type inequalities. We omit their proofs but refer the readers to Evans [8] for more details.

Theorem A.19 (Poincaré's inequality)**.** *Let U be a bounded, connected, and open subset of $\mathbb{R}^n$ with C^1 boundary ∂U. Assume $1 \leq p \leq \infty$. Then there exists a constant $C = C(n, p, U)$ depending only on n, p, and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \leq C(n, p, U)\|Du\|_{L^p(U)}$$

for each function $u \in W^{1,p}(U)$ where $(u)_U := \dfrac{1}{|U|}\displaystyle\int_U u\, dy$.

Theorem A.20 (Poincaré's inequality on balls)**.** *Assume $1 \leq p \leq \infty$. Then there exists a constant $C = C(n, p)$ depending only on n and p such that*

$$\|u - (u)_{x,r}\|_{L^p(B_r(x))} \leq C(n, p) \cdot r\|Du\|_{L^p(B_r(x))}$$

for each ball $B_r(x) \subset \mathbb{R}^n$ and each function $u \in W^{1,p}(B_r(x))$ where $(u)_{x,r} := \dfrac{1}{|B_r(x)|}\displaystyle\int_{B_r(x)} u\, dy$.

A simple application is the embedding of $W^{1,p}(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.

Corollary A.1. *Let $n \geq 1$ and suppose $u \in W^{1,n}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then $u \in BMO(\mathbb{R}^n)$.*

Proof. From Theorem A.20 with $p = 1$ and Hölder's inequality, we get

$$\begin{aligned}
\frac{1}{|B_r(x)|}\int_{B_r(x)} |u - (u)_{x,r}|\, dy &\leq Cr\frac{1}{|B_r(x)|}\int_{B_r(x)} |Du|\, dy \\
&\leq Cr\Big(\frac{1}{|B_r(x)|}\int_{B_r(x)} |Du|^n\, dy\Big)^{1/n} \\
&\leq C\Big(\int_{B_r(x)} |Du|^n\, dy\Big)^{1/n}.
\end{aligned}$$

Hence, we deduce that

$$\|u\|_{BMO(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u-(u)_{x,r}|\, dy \le C(n)\|u\|_{W^{1,n}(\mathbb{R}^n)}.$$

□

A.3 Convergence Theorems

Let $(X, \mathcal{A}, \mu)$ be a fixed measure space.

Theorem A.21 (Lebesgue's Montone Convergence)**.** *Let $\{f_n\}$ be a* **monotone increasing** *sequence of non-negative measurable functions that converges* **pointwise** *to a function $f(x)$, i.e.,*

(a) $0 \le f_1(x) \le f_2(x) \le \ldots \le f_n(x) \le \ldots \le \infty$ *for every $x \in X$ (monotone increasing),*

(b) and

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for every } x \in X \text{ (pointwise convergence).}$$

Then f is measurable and

$$\int_X f_n\, d\mu \longrightarrow \int_X f\, d\mu \quad \text{as } n \longrightarrow \infty.$$

Lemma A.1 (Fatou's)**.** *If $f_n : X \longrightarrow [0,\infty]$ is measurable, for each positive integer n, then*

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int_X f_n\, d\mu.$$

The next is a consequence of Fatou's lemma which we often use. For instance, it implies that strong solutions of elliptic equations on a bounded domain satisfy the equation pointwise almost everywhere in the domain.

Corollary A.2. *Suppose that f is a non-negative measurable function. Then $f = 0$ μ-almost everywhere in X if and only if*

$$\int_X f\, d\mu = 0. \tag{A.26}$$

Proof. If (A.26) holds, let

$$E_n = \left\{ x \in X \,\middle|\, f(x) > 1/n \right\},$$

so that $f \ge (1/n)\chi_{E_n}$, from which

$$0 = \int_X f\, d\mu \ge \frac{1}{n}\mu(E_n) \ge 0.$$

Thus, $\mu(E_n) = 0$ and so the set

$$\{x \in X \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$$

has measure zero by the countable additive property of measures.

Conversely, assume $f = 0$ μ-almost everywhere. If

$$E = \{x \in X \mid f(x) > 0\},$$

then obviously $\mu(E) = 0$. Then set $f_n = n\chi_E$ so that $f \leq \liminf f_n$. Thus, by Fatou's lemma,

$$0 \leq \int_X f \, d\mu \leq \liminf \int_X f_n \, d\mu = 0.$$

Hence, $\|f\|_{L^1(\mu)} = 0$, and this completes the proof. □

We can invoke the previous corollary to replace pointwise convergence with μ-almost everywhere convergence in Theorem A.21 but the limit function is assumed to be measurable a priori.

Corollary A.3. *Let $\{f_n\}$ be a* **monotone increasing** *sequence of non-negative measurable functions that converges μ-almost everywhere in X to a non-negative measurable function $f(x)$. Then*

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Proof. Choose $N \in \mathcal{A}$ be such that $\mu(N) = 0$ and $\{f_n\}$ converges to f at every point of $M = X \backslash N$. Then $\{f_n \chi_M\}$ converges to $f\chi_M$ in X. Thus Theorem A.21 implies that

$$\int_X f\chi_M \, d\mu = \lim_{n\to\infty} \int_X f_n \chi_M \, d\mu.$$

Since $\mu(N) = 0$, the functions $f\chi_N$ and $f_n\chi_N$ vanish μ-almost everywhere. It follows from Corollary A.26 that

$$\int_X f\chi_N \, d\mu = 0 \text{ and } \int_X f_n\chi_N \, d\mu = 0.$$

Since $f = f\chi_M + f\chi_N$ and $f_m = f_n\chi_M + f_n\chi_N$, it follows that

$$\int_X f \, d\mu = \int_X f\chi_M \, d\mu = \lim_{n\to\infty} \int_X f_n\chi_M \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

□

Theorem A.22 (Lebesgue's Dominated Convergence)**.** *Suppose $\{f_n\}$ is a sequence of measurable functions on X such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \le g(x) \quad \textit{for } n = 1, 2, 3, \ldots;\ x \in X,$$

then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu.$$

An immediate application of Theorem A.22 is the following

Corollary A.4. *If $t \longrightarrow f(x,t)$ is continuous on $[a,b]$ for each $x \in X$, and if there exists $g \in L^1(\mu)$ such that $|f(x,t)| \le g(x)$ for $x \in X$, then the function F defined by*

$$F(t) = \int_X f(x,t)\, d\mu(x) \tag{A.27}$$

is continuous for each t in $[a,b]$.

Another basic application of Theorem A.22 indicates when we may differentiate F and when it is equivalent to passing derivatives onto the integrand f. Hereafter, an integrable function f on X means f is a measurable function on X belonging to $L^1(\mu)$.

Corollary A.5. *Suppose that for some t_0 in $[a,b]$, the function $x \longrightarrow f(x,t_0)$ is integrable on X, that $\partial f/\partial t$ exists on $X \times [a,b]$, and that there exists an integrable function g on X such that*

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x).$$

Then the function F as defined in (A.27) *is differentiable on $[a,b]$ and*

$$\frac{dF}{dt}(t) = \frac{d}{dt}\int_X f(x,t)\, d\mu(x) = \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x).$$

Proof. Let t be any point of $[a,b]$. If $\{t_n\}$ is a sequence in $[a,b]$ converging to t with $t_n \ne t$, then

$$\frac{\partial f}{\partial t}(x,t) = \lim_{n\to\infty} \frac{f(x,t_n) - f(x,t)}{t_n - t},\ x \in X.$$

Therefore, the function $x \longrightarrow (\partial f/\partial t)(x,t)$ is measurable.

If $x \in X$ and $t \in [a, b]$, by the mean-value theorem, there exists s_1 between t_0 and t such that

$$f(x,t) - f(x,t_0) = (t - t_0)\frac{\partial f}{\partial t}(x, s_1).$$

Therefore,

$$|f(x,t)| \leq |f(x,t_0)| + |t - t_0| g(x),$$

which implies that the function $x \longrightarrow f(x,t)$ is integrable for each t in $[a, b]$. Hence, if $t_n \neq t$, then

$$\frac{F(t_n) - F(t)}{t_n - t} = \int_X \frac{f(x,t_n) - f(x,t)}{t_n - t}\, d\mu(x).$$

Since this integrand is dominated by $g(x)$, we may apply Theorem A.22 to conclude the desired result.

□

We can use Theorem A.22 to establish a similar convergence result in the Lebesgue spaces $L^p(\mu)$ with $1 \leq p < \infty$.

Theorem A.23. *Let $1 \leq p < \infty$ and suppose $\{f_n\}$ is a sequence in $L^p(\mu)$ which converges μ-almost everywhere to a measurable function f. If there exists a $g \in L^p(\mu)$ such that*

$$|f_n(x)| \leq g(x), \ x \in X, \ n \in N,$$

then f belongs to $L^p(\mu)$ and $\{f_n\}$ converges in L^p to f.

Proof. Assume $1 < p < \infty$ since the case $p = 1$ is exactly Theorem A.22. Obviously, the following two properties hold for μ-almost everywhere,

$$|f_n(x) - f(x)|^p \leq [2g(x)]^p, \ \text{ and } \ \lim_{n\to\infty} |f_n(x) - f(x)|^p = 0;$$

and there holds $[2g]^p$ and thus g^p belongs to $L^1(\mu)$. Hence, from Theorem A.22, we get

$$\lim_{n\to\infty} \int_X |f_n - f|^p \, d\mu = 0,$$

and this completes the proof of the theorem.

□

Remark A.6. *Lebesgue's dominated convergence theorem and its extension provide sufficient conditions that guarantee when pointwise convergence of a sequence of measurable functions implies strong convergence in the L^p norm topology; namely, if the sequence of functions can be compared to an L^p function, then pointwise convergence implies L^p convergence. Conversely, L^p convergence does not generally imply pointwise convergence. We give an example below illustrating this.*

Let $X = [0,1]$, the sigma algebra $\mathcal{A}$ are the Borel sets, and μ is the Lebesgue measure. Consider the ordered list of intervals

$[0,1], [0,\frac{1}{2}], [\frac{1}{2},1], [0,\frac{1}{3}], [\frac{1}{3},\frac{2}{3}], [\frac{2}{3},1], [0,\frac{1}{4}], [\frac{1}{4},\frac{1}{2}], [\frac{1}{2},\frac{3}{4}], [\frac{3}{4},1], [0,\frac{1}{5}], [\frac{1}{5},\frac{2}{5}], \ldots$; let f_n be the characteristic function of the n^{th} interval on this list, and let f be identically zero. If $n > m(m+1)/2 = 1+2+\ldots+m$, then f_n is a characteristic function of an interval I whose measure is at most $1/m$. Hence,

$$\|f - f\|^p_{L^p(\mu)} = \int_X |f_n - f|^p \, d\mu = \int_X |f_n|^p \, d\mu = \int_X f_n \, d\mu = \mu(I) \leq 1/m,$$

and this shows $\{f_n\}$ converges in L^p to $f \equiv 0$.

On the other hand, if x is any point of $[0,1]$, then the sequence of numbers $\{f_n(x)\}$ has a subsequence consisting only of 1's and another subsequence consisting of 0's. Therefore, the sequence $\{f_n\}$ does not converge at any point of $[0,1]$! (although we may select a particular subsequence of $\{f_n\}$ which does converge to f).

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